MIXED MULTIPLICITY SYSTEMS AND RELATED INVARIANTS

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ABSTRACT: This paper builds mixed multiplicity systems; the Euler-Poincare characteristic and the mixed multiplicity symbol of $\mathbb{N}^d$-graded modules with respect to a mixed multiplicity system, and proves that the Euler-Poincare characteristic and the mixed multiplicity symbol of mixed multiplicity systems of the type $(k_1, \ldots, k_d)$ and the mixed multiplicity of the type $(k_1, \ldots, k_d)$ are the same. As an application, we get results for mixed multiplicities of ideals.

1 Introduction

The mixed multiplicity is an important invariant of Algebraic Geometry and Commutative Algebra. Risler-Teissier [20] in 1973 showed that each mixed multiplicity of $\mathfrak{n}$-primary ideals (in a noetherian local ring with maximal ideal $\mathfrak{n}$) is the multiplicity of an ideal generated by a superficial sequence. For the case of arbitrary ideals, the first author [25] in 2000 described mixed multiplicities as the Samuel multiplicity via (FC)-sequences. And by using filter-regular sequences, Manh-Viet [31] in 2011 characterized mixed multiplicities of multigraded modules in terms of the length of modules. Moreover, in two recent papers, by a new approach, the authors gave the additivity and reduction formulas for mixed multiplicities [33] and formulas transmuting mixed multiplicities via rank of modules [34]. In past years, the theory of mixed multiplicities has attracted much attention and has been continually developed (see e.g. [4, 5, 10, 11, 12, 13, 14, 15, 16, 19, 22, 23, 27, 28, 29, 30, 32]). However, how to find mixed multiplicity formulas, which are analogous to Auslander-Buchsbaum's formula (see e.g. [1] or [3, Theorem 4.7.4]) and Serre's formula (see e.g. [17] or [3, Theorem 4.7.6]) for the Samuel multiplicity theory, is not yet known. And this problem became an open question of the mixed multiplicity theory.

This research was in part supported by a grant from NAFOSTED.
Mathematics Subject Classification (2010): Primary 13H15. Secondary 13A02, 13E05, 13E10, 14C17.
Key words and phrases: Mixed multiplicity, Euler-Poincare characteristic; Koszul complex.

Let $(A, \mathfrak{m})$ be an artinian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $A/\mathfrak{m}$. Denote by $\mathbb{N}$ the set of all the non-negative integers. Let d be a positive integer. Put $\mathbf{e}_i = (0, \ldots, \underset{(i)}{1}, \ldots, 0) \in \mathbb{N}^d$ for each $1 \le i \le d$ and $\mathbf{k} = (k_1, \ldots, k_d)$; $\mathbf{k}! = k_1! \cdots k_d!$; $\mid \mathbf{k} \mid = k_1 + \cdots + k_d$ for each $(k_1, \ldots, k_d) \in \mathbb{N}^d$. Moreover, set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{N}^d$; $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$ and $\mathbf{n}^{\mathbf{k}} = n_1^{k_1} \cdots n_d^{k_d}$ for each $\mathbf{n}, \mathbf{k} \in \mathbb{N}^d$ and $\mathbf{n} \ge \mathbf{1}$. Let $S = \bigoplus_{\mathbf{n} \in \mathbb{N}^d} S_{\mathbf{n}}$ be a finitely generated standard $\mathbb{N}^d$-graded algebra over A and let $M = \bigoplus_{\mathbf{n} \in \mathbb{N}^d} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^d$-graded S-module that satisfies $M = SM_{\mathbf{0}}$. For each subset $\mathbf{x}$ of S, we assign $\mathbf{x}M = 0$ if $\mathbf{x} = \emptyset$, and $\mathbf{x}M = (\mathbf{x})M$ if $\mathbf{x} \neq \emptyset$. Set $S_{++} = \bigoplus_{\mathbf{n} \ge \mathbf{1}} S_{\mathbf{n}}$ and $S_i = S_{\mathbf{e}_i}$ for $1 \le i \le d$. Denote by $\operatorname{Proj} S$ the set of the homogeneous prime ideals of S which do not contain S_{++}. Put

$$\operatorname{Supp}_{++} M = \{P \in \operatorname{Proj} S \mid M_P \neq 0\}.$$

Assume that $S_{\mathbf{1}} = S_{(1,\ldots,1)}$ is not contained in $\sqrt{\operatorname{Ann}_S M}$ and $\dim \operatorname{Supp}_{++} M = s$, then by [8, Theorem 4.1], $\ell_A[M_{\mathbf{n}}]$ is a polynomial of degree s for all large $\mathbf{n}$. Denote by $P_M(\mathbf{n})$ the Hilbert polynomial of the Hilbert function $\ell_A[M_{\mathbf{n}}]$. The terms of total degree s in the polynomial $P_M(\mathbf{n})$ have the form $\sum_{|\mathbf{k}| = s} e(M; \mathbf{k}) \dfrac{\mathbf{n}^{\mathbf{k}}}{\mathbf{k}!}$. Then $e(M; \mathbf{k})$ are non-negative integers not all zero, called the *mixed multiplicity of M of the type $\mathbf{k}$* [8]. Now for each $\mathbf{k} \in \mathbb{N}^d$ such that $\mid \mathbf{k} \mid \ge \dim \operatorname{Supp}_{++} M$, we put

$$E(M; \mathbf{k}) = \begin{cases} e(M; \mathbf{k}) & \text{if } \mid \mathbf{k} \mid = \dim \operatorname{Supp}_{++} M, \\ 0 & \text{if } \mid \mathbf{k} \mid > \dim \operatorname{Supp}_{++} M. \end{cases}$$

Let $(R, \mathfrak{n})$ be a noetherian local ring with maximal ideal $\mathfrak{n}$ and infinite residue field $R/\mathfrak{n}$. Let $J, I_1, \ldots, I_d$ be ideals of R with J being $\mathfrak{n}$-primary. Let N be a finitely generated R-module. Set $I = JI_1 \cdots I_d$; $\mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n \ge 0} (\mathfrak{a} : \mathfrak{b}^n)$; and $\mathbf{I} = I_1, \ldots, I_d$; $\mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \ldots, I_d^{[k_d]}$ and $\mathbb{I}^{\mathbf{n}} = I_1^{n_1} \cdots I_d^{n_d}$. We get an $\mathbb{N}^{(d+1)}$-graded algebra and an $\mathbb{N}^{(d+1)}$-graded module:

$$T = \bigoplus_{n \ge 0,\, \mathbf{n} \ge \mathbf{0}} \frac{J^n \mathbb{I}^{\mathbf{n}}}{J^{n+1} \mathbb{I}^{\mathbf{n}}} \text{ and } \mathcal{N} = \bigoplus_{n \ge 0,\, \mathbf{n} \ge \mathbf{0}} \frac{J^n \mathbb{I}^{\mathbf{n}} N}{J^{n+1} \mathbb{I}^{\mathbf{n}} N},$$

respectively. Note that T is a finitely generated standard $\mathbb{N}^{(d+1)}$-graded algebra over an artinian local ring R/J and $\mathcal{N}$ is a finitely generated $\mathbb{N}^{(d+1)}$-graded T-module; and $\dim \operatorname{Supp}_{++} \mathcal{N} = \dim \dfrac{N}{0_N : I^\infty} - 1$ by [8] and [25](see [14]). The mixed multiplicity of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ is denoted by $e\big(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N\big)$ and which is called the *mixed multiplicity of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0 + 1, \mathbf{k})$ (see [8, 14, 24]).

In this paper, we first build mixed multiplicity systems; the Euler-Poincare characteristic and the mixed multiplicity symbol of finitely generated $\mathbb{N}^d$-graded S-modules with respect to a mixed multiplicity system.

Definition 3.2. Let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements in $\bigcup_{j=1}^{d} S_j$ consisting of m_1 elements of $S_1, \ldots, m_d$ elements of S_d. Then $\mathbf{x}$ is called a *mixed multiplicity system of M of the type* $\mathbf{m} = (m_1, \ldots, m_d)$ if $\dim \operatorname{Supp}_{++}\left(\dfrac{M}{\mathbf{x}M}\right) \leq 0.$

In the case that $\mathbf{x}$ is a mixed multiplicity system of the type $\mathbf{m}$ and $\mid \mathbf{m} \mid = n$, it can be verified that $\sum_{i=0}^{n}(-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}]$ is a constant for $\mathbf{n} \gg 0$ (see Remark 3.3). And denote by $\chi(\mathbf{x}, M)$ this constant. Then $\chi(\mathbf{x}, M)$ is briefly called the *Euler-Poincare characteristic of M with respect to* $\mathbf{x}$.

Definition 3.7. Let $\mathbf{x} = x_1, \ldots, x_n$ be a mixed multiplicity system of M. If $n = 0$, then $\ell_A[M_{\mathbf{n}}] = c$ (const) for all $\mathbf{n} \gg 0$ and we set $\widetilde{e}(\mathbf{x}, M) = c$. If $n > 0$, we set $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/x_1) - \widetilde{e}(\mathbf{x}', 0_M : x_1)$, here $\mathbf{x}' = x_2, \ldots, x_n$. We call $\widetilde{e}(\mathbf{x}, M)$ the *mixed multiplicity symbol of M with respect to* $\mathbf{x}$.

As one might expect, we proved that the Euler-Poincare characteristic and the mixed multiplicity symbol of mixed multiplicity systems of the type $\mathbf{k} = (k_1, \ldots, k_d)$ and the mixed multiplicity of the type $\mathbf{k}$ are the same by the following result.

Theorem 3.12. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{x}$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Assume that* $\dim \operatorname{Supp}_{++}M \leq \mid \mathbf{k} \mid$. *Then we have*

$$E(M; \mathbf{k}) = \chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M).$$

This theorem shows that the mixed multiplicity symbol $\widetilde{e}(\mathbf{x}, M)$ of M with respect to a mixed multiplicity system $\mathbf{x}$ does not depend on the order of the elements of $\mathbf{x}$ and the Euler-Poincare characteristic $\chi(\mathbf{x}, M)$ depends only on the type of $\mathbf{x}$. These properties seem hard to be found by the definitions of them.

Theorem 3.12 not only yields important consequences in the case of graded modules (see e.g. Corollary 3.13; Corollary 3.15; Theorem 3.16; Corollary 3.18; Corollary 3.19 and Corollary 3.20) but also gives interesting applications to mixed multiplicities of ideals as the following preliminary facts.

Let us begin with defining a system of elements $\mathbf{x} = x_1, \ldots, x_s$ in R such that the image $\mathbf{x}^*$ of $\mathbf{x}$ in T is a mixed multiplicity system of $\mathcal{N}$. Then we need to choose $\mathbf{x}$

satisfying the properties:

$$[\mathcal{N}/(x_1^*,\ldots,x_i^*)\mathcal{N}]_{(m,\,\mathbf{m})} \cong \Big[\bigoplus_{n\geq 0,\,\mathbf{n}\geq \mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}(N/(x_1,\ldots,x_i)N)}{J^{n+1}\mathbb{I}^{\mathbf{n}}(N/(x_1,\ldots,x_i)N)}\Big]_{(m,\,\mathbf{m})}$$

for all $m \gg 0$; $\mathbf{m} \gg \mathbf{0}$ and $0 \leq i \leq s$, and moreover,

$$\dim \operatorname{Supp}_{++}\frac{\mathcal{N}}{\mathbf{x}^*\mathcal{N}} = \dim \frac{N}{\mathbf{x}N : I^\infty} - 1 \leq 0.$$

That is why we use the following sequences (see Section 4).

Definition 4.2. An element $a \in R$ is called a *Rees's superficial element* of N with respect to $\mathbf{I}$ if there exists $i \in \{1,\ldots,d\}$ such that $a \in I_i$ and $aN \bigcap \mathbb{I}^{\mathbf{n}} I_i N = a\mathbb{I}^{\mathbf{n}}N$ for all $\mathbf{n} \gg 0$. Let $x_1,\ldots,x_t$ be a sequence in R. Then $x_1,\ldots,x_t$ is called a *Rees's superficial sequence* of N with respect to $\mathbf{I}$ if x_{j+1} is a Rees's superficial element of $N/(x_1,\ldots,x_j)N$ with respect to $\mathbf{I}$ for all $j = 0,1,\ldots,t-1$. A Rees's superficial sequence of N consisting of k_1 elements of $I_1,\ldots,k_d$ elements of I_d is called a Rees's superficial sequence of N of the type $\mathbf{k} = (k_1,\ldots,k_d)$.

Definition 4.4. Let $\mathbf{x}$ be a Rees's superficial sequence of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0,\mathbf{k})$. Then $\mathbf{x}$ is called a *mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0,\mathbf{k})$ if $\dim \dfrac{N}{\mathbf{x}N : I^\infty} \leq 1$.

Under our point of view in this paper, both the conditions of mixed multiplicity systems of N are necessary to characterize mixed multiplicities of ideals in terms of the Euler-Poincare characteristic and the mixed multiplicity symbol of $\mathcal{N}$ with respect to a mixed multiplicity system of $\mathcal{N}$.

As an application of Theorem 3.12 we obtain the following theorem.

Theorem 4.9. *Let $(R,\mathfrak{n})$ denote a noetherian local ring with maximal ideal $\mathfrak{n}$ and infinite residue field $R/\mathfrak{n}$. Let $J, I_1,\ldots,I_d$ be ideals of R with J being $\mathfrak{n}$-primary. Let N be a finitely generated R-module. Assume that $I = JI_1\cdots I_d$ is not contained in $\sqrt{\mathrm{Ann}_R N}$ and $(k_0,\mathbf{k}) \in \mathbb{N}^{d+1}$ such that $k_0 + \mid \mathbf{k} \mid = \dim \dfrac{N}{0_N : I^\infty} - 1$. Let $\mathbf{x}$ be a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0,\mathbf{k})$ and $\mathbf{x}^*$ the image of $\mathbf{x}$ in $\bigcup_{i=0}^d T_i$. Then we have*

$$e(J^{[k_0+1]},\mathbf{I}^{[\mathbf{k}]};N) = \chi(\mathbf{x}^*,\mathcal{N}) = \widetilde{e}(\mathbf{x}^*,\mathcal{N}).$$

In order to continue describing mixed multiplicity formulas of ideals we need the next notation.

Let $k_0 \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^d$ such that $k_0 + \mid \mathbf{k} \mid \geq \dim \dfrac{N}{0_N : I^\infty} - 1$. We assign

$$E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \begin{cases} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) & \text{if } k_0 + \mid \mathbf{k} \mid = \dim \dfrac{N}{0_N : I^\infty} - 1, \\ 0 & \text{if } k_0 + \mid \mathbf{k} \mid > \dim \dfrac{N}{0_N : I^\infty} - 1. \end{cases}$$

Then from corollaries of Theorem 3.12 we get the following result for mixed multiplicities of ideals.

Corollary 4.13. *Assume that* $I = JI_1 \cdots I_d$ *is not contained in* $\sqrt{\mathrm{Ann}_R N}$ *and* $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}, N) \neq 0$. *Let* $\mathbf{x} = x_1, \ldots, x_s$ *be a mixed multiplicity system of* N *with respect to ideals* $J, \mathbf{I}$ *of the type* $(k_0, \mathbf{k})$. *Denote by* $(m_i, \mathbf{h}_i) = (m_i, h_{i1}, \ldots, h_{id})$ *the type of a subsequence* $x_1, \ldots, x_i$ *of* $\mathbf{x}$ *for each* $1 \leq i \leq s$. *And for each* $1 \leq i \leq s$, *set*

$$N_i = \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N}.$$

Then the following statements hold.

(i) $\dim \dfrac{N}{(x_1, \ldots, x_i)N : I^\infty} = \dim \dfrac{N}{0_N : I^\infty} - i$ *for each* $1 \leq i \leq s$.

(ii) $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \dfrac{N}{\mathbf{x}N : I^\infty}\big) - \sum_{i=1}^{s} E\big(J^{[k_0 - m_i + 1]}, \mathbf{I}^{[\mathbf{k} - \mathbf{h}_i]}; N_i\big)$.

The paper supplied the mixed multiplicity theory with the important formulas which are analogous to that in the Samuel multiplicity theory. We hope that these results will be results with potentialities. Our approach is based on ideas of Auslander and Buchsbaum in [1], which were also introduced in [3] and several different books. We need to choose exactly objects suitable for the goal of the paper.

This paper is divided into four sections. Section 2 is devoted to the discussion of mixed multiplicities of multigraded modules and filter-regular sequences that will be used in this paper. Moreover, we obtain an improvement version of [31, Theorem 3.4](see Theorem 2.4). In Section 3, we build mixed multiplicity systems; the Euler-Poincare characteristic and the mixed multiplicity symbol, and prove that the Euler-Poincare characteristic and the mixed multiplicity symbol of mixed multiplicity systems of the type $\mathbf{k}$ and the mixed multiplicity of the type $\mathbf{k}$ are the same (see Proposition 3.8; Proposition 3.10; Theorem 3.12 and Theorem 3.16). We also obtain profound results in some particular cases. Section 4 gives applications of Section 3 to mixed multiplicities of ideals and yields interesting consequences (see Corollary 4.8; Theorem 4.9; Corollary 4.12; Corollary 4.13; Corollary 4.14).

2 Filter-Regular Sequences and Mixed Multiplicities

This section defines mixed multiplicities and filter-regular sequences of finitely generated $\mathbb{N}^d$-graded modules; gives an improvement version of [31, Theorem 3.4] and a difference formula of the Hilbert polynomial $P_M(\mathbf{n})$ (Proposition 2.6).

Recall that $S_{\mathbf{1}} = S_{(1,\ldots,1)}$; $\mathbf{k} = (k_1, \ldots, k_d)$; $\mathbf{k}! = k_1! \cdots k_d!$; $\mid \mathbf{k} \mid = k_1 + \cdots + k_d$; $\mathbf{n}^{\mathbf{k}} = n_1^{k_1} \cdots n_d^{k_d}$; $\mathbf{e}_i = (0, \ldots, \underset{(i)}{1}, \ldots, 0) \in \mathbb{N}^d$ for each $1 \leq i \leq d$. Assume that $S_{\mathbf{1}} \nsubseteq \sqrt{\mathrm{Ann}_S M}$ and $\dim \mathrm{Supp}_{++} M = s$, then by [8, Theorem 4.1], $\ell_A[M_{\mathbf{n}}]$ is a polynomial of degree s for all large $\mathbf{n}$. Denote by $P_M(\mathbf{n})$ the Hilbert polynomial of the Hilbert function $\ell_A[M_{\mathbf{n}}]$. The terms of total degree s in the polynomial $P_M(\mathbf{n})$ have the form

$$\sum_{|\mathbf{k}| = s} e(M; \mathbf{k}) \frac{\mathbf{n}^{\mathbf{k}}}{\mathbf{k}!}.$$

Then $e(M; \mathbf{k})$ are non-negative integers not all zero, called the *mixed multiplicity of* M *of the type* $\mathbf{k}$ [8].

Denote by $\triangle^{\mathbf{k}} f(\mathbf{n})$ the $\mathbf{k}$-difference of the polynomial $f(\mathbf{n})$ for each $\mathbf{k} \in \mathbb{N}^d$. Then we have some following comments.

Remark 2.1. Assume that $\dim \mathrm{Supp}_{++} M = s \geq 0$. Since

$$P_M(\mathbf{n}) = \sum_{|\mathbf{k}| = s} e(M; \mathbf{k}) \frac{\mathbf{n}^{\mathbf{k}}}{\mathbf{k}!} + Q_M(\mathbf{n})$$

with $\deg Q_M(\mathbf{n}) < s$, it follows that $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = e(M; \mathbf{k})$ for all $\mathbf{k} \in \mathbb{N}^d$ satisfying $\mid \mathbf{k} \mid = s$. If we assign $\dim \mathrm{Supp}_{++} M = -\infty$ to the case that $\mathrm{Supp}_{++} M = \emptyset$ and the degree $-\infty$ to the zero polynomial then by [8, Theorem 4.1] and [31, Proposition 2.7], we always have

$$\deg P_M(\mathbf{n}) = \dim \mathrm{Supp}_{++} M.$$

In a recently appeared paper [31], by using S_{++}-filter-regular sequences, Manh-Viet answered to the question when mixed multiplicities of multigraded modules are positive and characterized these mixed multiplicities in terms of the length of modules (see [31, Theorem 3.4]).

We turn now to filter-regular sequences of multigraded modules. The notion of filter-regular sequences was introduced by Stuckrad and Vogel in [18](see [2]). The theory of filter-regular sequences became an important tool to study some classes of singular rings and has been continually developed (see e.g. [2, 9, 21, 22, 31, 33]).

Note that in [31] one defined S_{++}-filter-regular sequences in the condition $S_{\mathbf{1}} \nsubseteq \sqrt{\mathrm{Ann}_S M}$. In this paper, omitting the condition $S_{\mathbf{1}} \nsubseteq \sqrt{\mathrm{Ann}_S M}$, we need the following sequences.

Definition 2.2. Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $a \in S$ be a homogeneous element. Then a is called an *S_{++}-filter-regular element with respect to M* if $(0_M : a)_{\mathbf{n}} = 0$ for all large $\mathbf{n}$. Let $x_1, \ldots, x_t$ be homogeneous elements in S. We call that $x_1, \ldots, x_t$ is an *S_{++}-filter-regular sequence with respect to M* if x_i is an S_{++}-filter-regular element with respect to $\dfrac{M}{(x_1, \ldots, x_{i-1})M}$ for all $i = 1, \ldots, t$.

Remark 2.3. Since $(0_M : S_{++}^{\infty})_{\mathbf{n}} = 0$ for all large $\mathbf{n}$ by [31, Remark 2.4], it is easy to see that a homogeneous element $a \in S$ is an S_{++}-filter-regular element with respect to M if and only if $0_M : a \subseteq 0_M : S_{++}^{\infty}$. Moreover, for each $(k_1, \ldots, k_d) \in \mathbb{N}^d$ there exists an S_{++}-filter-regular sequence $\mathbf{x}$ in $\bigcup_{i=1}^d S_i$ with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d by [31, Proposition 2.2 and Note (ii)]. In this case, $\mathbf{x}$ is called an S_{++}-filter-regular sequence of the type $\mathbf{k} = (k_1, \ldots, k_d)$.

Using S_{++}-filter-regular sequences and [31, Remark 2.6], we obtain the following improvement version of [31, Theorem 3.4].

Theorem 2.4 (see [31, Theorem 3.4])**.** *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module such that $S_{\mathbf{1}} \nsubseteq \sqrt{\mathrm{Ann}_S M}$. Let $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $\mid \mathbf{k} \mid= \dim \mathrm{Supp}_{++} M$. Assume that $\mathbf{x}$ is an S_{++}-filter-regular sequence with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. Then we have*

$$e(M; \mathbf{k}) = \ell_A \left(\frac{M}{\mathbf{x}M} \right)_{\mathbf{n}}$$

for all large $\mathbf{n}$. And $e(M; \mathbf{k}) \neq 0$ if and only if $\dim \mathrm{Supp}_{++} \left(\frac{M}{\mathbf{x}M} \right) = 0$.

Proof. First, note that if $a \in S_i$ satisfies $(0_M : a)_{\mathbf{n}} = 0$ for all large $\mathbf{n}$, then

$$\ell_A[(M/aM)_{\mathbf{n}}] = \ell_A[M_{\mathbf{n}}] - \ell_A[M_{\mathbf{n}-\mathbf{e}_i}]$$

for all large $\mathbf{n}$ by [31, Remark 2.6]. Therefore $P_{M/aM}(\mathbf{n}) = P_M(\mathbf{n}) - P_M(\mathbf{n} - \mathbf{e}_i)$. So $\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n})$. Hence $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = P_{M/\mathbf{x}M}(\mathbf{n})$ since $\mathbf{x}$ is an S_{++}-filter-regular sequence with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements

of S_d. Since $|\mathbf{k}| = \dim \operatorname{Supp}_{++} M = \deg P_M(\mathbf{n})$, it follows that

$$\dim \operatorname{Supp}_{++}\left(\frac{M}{\mathbf{x}M}\right) = \deg P_{M/\mathbf{x}M}(\mathbf{n}) \leq 0$$

and $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = e(M;\mathbf{k})$ by Remark 2.1. Consequently,

$$e(M;\mathbf{k}) = P_{M/\mathbf{x}M}(\mathbf{n}) = \ell_A\left(\frac{M}{\mathbf{x}M}\right)_{\mathbf{n}}$$

for all $\mathbf{n} \gg 0$ and $e(M,\mathbf{k}) \neq 0$ if and only if $\dim \operatorname{Supp}_{++}\left(\dfrac{M}{\mathbf{x}M}\right) = 0$. □

Remark 2.5. From the proof of Theorem 2.4, it follows that if $a \in S_i$ is an S_{++}-filter-regular element with respect to M then $\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n})$. Hence

$$\dim \operatorname{Supp}_{++}\left(\frac{M}{aM}\right) \leq \dim \operatorname{Supp}_{++} M - 1.$$

Moreover, we have the following proposition.

Proposition 2.6. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $a \in S_i$. Then*

$$\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n}) - P_{(0_M:\,a)}(\mathbf{n} - \mathbf{e}_i).$$

Proof. Since the following short exact sequences

$$0 \longrightarrow aM_{\mathbf{n}-\mathbf{e}_i} \longrightarrow M_{\mathbf{n}} \longrightarrow \frac{M_{\mathbf{n}}}{aM_{\mathbf{n}-\mathbf{e}_i}} \longrightarrow 0$$

and

$$0 \longrightarrow (0_M : a)_{\mathbf{n}-\mathbf{e}_i} \longrightarrow M_{\mathbf{n}-\mathbf{e}_i} \xrightarrow{a} aM_{\mathbf{n}-\mathbf{e}_i} \longrightarrow 0,$$

which follow that $\ell_A[M_{\mathbf{n}}] - \ell_A[M_{\mathbf{n}-\mathbf{e}_i}] = \ell_A\left[\dfrac{M_{\mathbf{n}}}{aM_{\mathbf{n}-\mathbf{e}_i}}\right] - \ell_A[(0_M : a)_{\mathbf{n}-\mathbf{e}_i}]$. Remember that $\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = \ell_A[M_{\mathbf{n}}] - \ell_A[M_{\mathbf{n}-\mathbf{e}_i}]$ and $P_{M/aM}(\mathbf{n}) = \ell_A\left[\dfrac{M_{\mathbf{n}}}{aM_{\mathbf{n}-\mathbf{e}_i}}\right]$; and

$$P_{(0_M:a)}(\mathbf{n} - \mathbf{e}_i) = \ell_A[(0_M : a)_{\mathbf{n}-\mathbf{e}_i}]$$

for all $\mathbf{n} \gg 0$. Therefore $\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n}) - P_{(0_M:a)}(\mathbf{n} - \mathbf{e}_i)$. □

3 The Euler-Poincare Characteristic and Multiplicities

In this section, we first build mixed multiplicity systems; the Euler-Poincare characteristic and the mixed multiplicity symbol of finitely generated $\mathbb{N}^d$-graded S-modules with respect to a mixed multiplicity system. Next we prove that the Euler-Poincare characteristic and the mixed multiplicity symbol of mixed multiplicity systems of the type $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and the mixed multiplicity of the type $\mathbf{k}$ are the same, and give interesting consequences.

Let $\mathbf{x}$ be a system of n homogeneous elements in S. Then by [3, Remark 1.6.15], we may consider the following Koszul complex $K_\bullet(\mathbf{x}, M)$ of M with respect to $\mathbf{x}$ as an $\mathbb{N}^d$-graded complex with a differential of degree $\mathbf{0}$:

$$0 \longrightarrow K_n(\mathbf{x}, M) \longrightarrow K_{n-1}(\mathbf{x}, M) \longrightarrow \cdots \longrightarrow K_1(\mathbf{x}, M) \longrightarrow K_0(\mathbf{x}, M) \longrightarrow 0.$$

Denote by $H_\bullet(\mathbf{x}, M)$ the homology of the Koszul complex of M with respect to $\mathbf{x}$. We get the following sequence of the homology modules

$$H_\bullet(\mathbf{x}, M) : \ldots H_0(\mathbf{x}, M), H_1(\mathbf{x}, M), \ldots, H_n(\mathbf{x}, M) \ldots.$$

The Koszul complex theory became an important tool to study several different theories of Commutative Algebra and Algebraic Geometry.

Remark 3.1. Note that $K_i(\mathbf{x}, M)$ and $H_i(\mathbf{x}, M)$ are finitely generated $\mathbb{N}^d$-graded S-modules. And $\mathbf{x}H_i(\mathbf{x}, M) = 0$ for all $0 \leq i \leq n$ (see [3, Proposition 1.6.5(b)]). Moreover, we have the following notes.

(i) Since $\sqrt{\mathrm{Ann}_S(M/\mathbf{x}M)} = \sqrt{\mathrm{Ann}_S M + (\mathbf{x})}$ and $\mathrm{Ann}_S M + (\mathbf{x}) \subseteq \mathrm{Ann}_S H_i(\mathbf{x}, M)$ for $0 \leq i \leq n$, it follows that $\mathrm{Ann}_S\left(\dfrac{M}{\mathbf{x}M}\right) \subseteq \sqrt{\mathrm{Ann}_S H_i(\mathbf{x}, M)}$ for $0 \leq i \leq n$. Consequently, if $\dim \mathrm{Supp}_{++}\left(\dfrac{M}{\mathbf{x}M}\right) \leq 0$ then $\dim \mathrm{Supp}_{++} H_i(\mathbf{x}, M) \leq 0$ for all $0 \leq i \leq n$.

(ii) Assume that $\dim \mathrm{Supp}_{++}\left(\dfrac{M}{\mathbf{x}M}\right) \leq 0$. Then for each $0 \leq i \leq n$ we have $\dim \mathrm{Supp}_{++} H_i(\mathbf{x}, M) \leq 0$ by (i). Therefore by Remark 2.1, we get

$$\ell_A[H_i(\mathbf{x}, M)_\mathbf{n}] = a_i(\text{const})$$

for all $\mathbf{n} \gg 0$ and for each $0 \leq i \leq n$. Hence there exists $\mathbf{v} \in \mathbb{N}^d$ such that for each $0 \leq i \leq n$, $\ell_A[H_i(\mathbf{x}, M)_\mathbf{n}] = a_i$ for all $\mathbf{n} \geq \mathbf{v}$. Consequently, we get the

following constant

$$\chi(\mathbf{x}, M) = \sum_{i=0}^{n} (-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}]$$

for all $\mathbf{n} \geq \mathbf{v}$. In this case, $\chi(\mathbf{x}, M)$ is briefly called the *Euler-Poincare characteristic of M with respect to* $\mathbf{x}$.

The Euler-Poincare characteristic of M with respect to a system $\mathbf{x}$ is one of the most important invariants of this paper. But the existence of the Euler-Poincare characteristic depends on $\mathbf{x}$. Defining a sequence $\mathbf{x}$ such that there exists the Euler-Poincare characteristic $\chi(\mathbf{x}, M)$ and $\chi(\mathbf{x}, M)$ characterizes some mixed multiplicity of M is the reason why we give the following concept.

Definition 3.2. Let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements in $\bigcup_{j=1}^{d} S_j$ consisting of m_1 elements of $S_1, \ldots, m_d$ elements of S_d. Then $\mathbf{x}$ is called a *mixed multiplicity system of M of the type* $\mathbf{m} = (m_1, \ldots, m_d)$ if $\dim \operatorname{Supp}_{++}\left(\dfrac{M}{\mathbf{x}M}\right) \leq 0$.

Remember that in the case of modules over local rings, one gave the multiplicity systems that generate the ideals of definition. The results of this paper will show the usefulness of mixed multiplicity systems.

Remark 3.3. We have the following comments for mixed multiplicity systems.

(i) By Remark 2.3, for each $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ there exists an S_{++}-filter-regular sequence $\mathbf{x}$ in $\bigcup_{i=1}^{d} S_i$ with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. Now we choose $\mathbf{k}$ such that $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M$. Then by Remark 2.5, we get $\dim \operatorname{Supp}_{++}\left(\dfrac{M}{\mathbf{x}M}\right) \leq \dim \operatorname{Supp}_{++} M - \mid \mathbf{k} \mid \leq 0$. Hence $\mathbf{x}$ is a mixed multiplicity system. So for any $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M$, there exists a mixed multiplicity system of M of the type $\mathbf{k}$.

(ii) Let $\mathbf{x} = x_1, \ldots, x_n$ be a mixed multiplicity system of M of the type $\mathbf{m}$ and $x_1 \in S_i$. Then it can be verified that $\mathbf{x}' = x_2, \ldots, x_n$ is a mixed multiplicity system of the type $\mathbf{m} - \mathbf{e}_i$ of both M/x_1M and $(0_M : x_1)$. And from Remark 3.1(ii), it follows that there exists the Euler-Poincare characteristic $\chi(\mathbf{x}, M)$ and $\chi(\mathbf{x}, M) = \sum_{i=0}^{n} (-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}]$ for all $\mathbf{n} \gg 0$, here $n = \mid \mathbf{m} \mid$. Note that if $n = 0$ then $\chi(\mathbf{x}, M) = \chi(\emptyset, M) = \ell_A[M_{\mathbf{n}}]$ for all $\mathbf{n} \gg 0$.

Lemma 3.4. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$ *be an exact sequence of* $\mathbb{N}^d$*-graded* S*-modules. And let* $\mathbf{x}$ *be a sequence of elements in* $\bigcup_{i=1}^d S_i$*. Then* $\mathbf{x}$ *is a mixed multiplicity system of* M *if and only if* $\mathbf{x}$ *is a mixed multiplicity system of both* M' *and* $M"$*.*

Proof. Note that $\sqrt{\mathrm{Ann}(F/\mathfrak{c}F)} = \sqrt{\mathrm{Ann}F + \mathfrak{c}}$ for each ideal $\mathfrak{c}$ and each module F. Hence since $\mathrm{Ann}_S M \subseteq \mathrm{Ann}_S M'$, it follows that

$$\dim \mathrm{Supp}_{++}\left(\frac{M'}{\mathbf{x}M'}\right) \le \dim \mathrm{Supp}_{++}\left(\frac{M}{\mathbf{x}M}\right).$$

On the other hand from the exactness of

$$M'/\mathbf{x}M' \xrightarrow{g} M/\mathbf{x}M \to M"/\mathbf{x}M" \to 0,$$

it implies that $\dim \mathrm{Supp}_{++}\left(\frac{M'}{\mathbf{x}M'}\right) \ge \dim \mathrm{Supp}_{++}\mathrm{Im}\; g$ and

$$\dim \mathrm{Supp}_{++}\left(\frac{M}{\mathbf{x}M}\right) = \max\left\{ \dim \mathrm{Supp}_{++}\mathrm{Im}\; g,\ \dim \mathrm{Supp}_{++}\left(\frac{M"}{\mathbf{x}M"}\right)\right\}.$$

From the above facts, we immediately get that $\dim \mathrm{Supp}_{++}\left(\frac{M}{\mathbf{x}M}\right) \le 0$ if and only if $\dim \mathrm{Supp}_{++}\left(\frac{M'}{\mathbf{x}M'}\right) \le 0$ and $\dim \mathrm{Supp}_{++}\left(\frac{M"}{\mathbf{x}M"}\right) \le 0$. Thus, $\mathbf{x}$ is a mixed multiplicity system of M if and only if $\mathbf{x}$ is a mixed multiplicity system of both M' and $M"$. $\square$

Some basic properties of the Euler-Poincare characteristic with respect to mixed multiplicity systems are stated by the following lemma.

Lemma 3.5. *Let* $\mathbf{x} = x_1, \ldots, x_n$ *be a mixed multiplicity system of* M*. Then the following statements hold.*

(i) $\chi(\mathbf{x}, \_\,)$ *is additive on short exact sequences, i.e., if*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

is a short exact sequence then $\chi(\mathbf{x}, M) = \chi(\mathbf{x}, M') + \chi(\mathbf{x}, M")$*.*

(ii) *If* $x_1 \in \sqrt{\mathrm{Ann}_S M}$ *then* $\chi(\mathbf{x}, M) = 0$*.*

(iii) *If* x_1 *is* M*-regular and* $\mathbf{x}' = x_2, \ldots, x_n$ *then* $\chi(\mathbf{x}, M) = \chi(\mathbf{x}', M/x_1M)$*.*

Proof. Without loss of generality, we may assume that $\mathbb{N}^d$-graded S-homomorphisms in this proof are $\mathbb{N}^d$-graded S-homomorphisms of degree $\mathbf{0}$.

The proof of (i): Since the Koszul complex is an exact functor, the exact sequence of $\mathbb{N}^d$-graded S-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

yields a long exact sequence

$$\cdots \longrightarrow H_i(\mathbf{x}, M') \longrightarrow H_i(\mathbf{x}, M) \longrightarrow H_i(\mathbf{x}, M") \longrightarrow H_{i-1}(\mathbf{x}, M') \longrightarrow \cdots$$

of $\mathbb{N}^d$-graded homology S-modules $H_\bullet(\mathbf{x}, M)$. Hence for each $\mathbf{n} \in \mathbb{N}^d$ we have an exact sequence of A-modules

$$\cdots \longrightarrow H_i(\mathbf{x}, M')_{\mathbf{n}} \longrightarrow H_i(\mathbf{x}, M)_{\mathbf{n}} \longrightarrow H_i(\mathbf{x}, M")_{\mathbf{n}} \longrightarrow H_{i-1}(\mathbf{x}, M')_{\mathbf{n}} \longrightarrow \cdots .$$

Recall that $\mathbf{x}$ is also a mixed multiplicity system of both M' and $M"$ by Lemma 3.4. Hence by the addivity of length, the alternating sum of the lengths of the modules in this exact sequence is zero. This fact follows (i).

The proof of (ii): First we consider the case that $x_1 \in \mathrm{Ann}_S M$. Recall that $\mathbf{x}' = x_2, \ldots, x_n$. By [3, Corollary 1.6.13(a)], we have the following exact sequence

$$\cdots \xrightarrow{\pm x_1} H_i(\mathbf{x}', M) \longrightarrow H_i(\mathbf{x}, M) \longrightarrow H_{i-1}(\mathbf{x}', M) \xrightarrow{\pm x_1} H_{i-1}(\mathbf{x}', M) \longrightarrow \cdots . \quad (3.1)$$

Since $x_1 M = 0$, $\mathbf{x}'$ is also a mixed multiplicity system of M. Moreover, x_1 annihilates $H_i(\mathbf{x}', M)$ for all i. Hence the exact sequence (3.1) breaks up into exact sequences

$$0 \longrightarrow H_i(\mathbf{x}', M) \longrightarrow H_i(\mathbf{x}, M) \longrightarrow H_{i-1}(\mathbf{x}', M) \longrightarrow 0$$

for all i. Therefore

$$\ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}] = \ell_A[H_i(\mathbf{x}', M)_{\mathbf{n}}] + \ell_A[H_{i-1}(\mathbf{x}', M)_{\mathbf{n}}]$$

for all i and for all large $\mathbf{n}$. Note that $H_{n+1}(\mathbf{x}, M) = H_{-1}(\mathbf{x}, M) = 0$,

$$\sum_{i=0}^{n+1} (-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}] = \sum_{i=0}^{n+1} (-1)^i \Big[\ell_A[H_i(\mathbf{x}', M)_{\mathbf{n}}] + \ell_A[H_{i-1}(\mathbf{x}', M)_{\mathbf{n}}]\Big] = 0 \quad (3.2)$$

for all large $\mathbf{n}$. Since $\ell_A[H_{n+1}(\mathbf{x}, M)_{\mathbf{n}}] = 0$ and

$$\chi(\mathbf{x}, M) = \sum_{i=0}^{n} (-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}]$$

for all $\mathbf{n} \gg 0$, $\chi(\mathbf{x}, M) = 0$ by (3.2). So if $x_1 M = 0$ then $\chi(\mathbf{x}, M) = 0$. From this it follows that $\chi(\mathbf{x}, M/aM) = 0$ for all $a \in \mathbf{x}$.

We turn now to the case that $x_1 \in \sqrt{\mathrm{Ann}_S M}$. Then there exists $u > 0$ such that $x_1^u M = 0_M$. On the other hand since $\chi(\mathbf{x}, M/x_1M) = 0$ and the exact sequence

$$0 \longrightarrow x_1 M \longrightarrow M \longrightarrow M/x_1M \longrightarrow 0,$$

which follow by (i) that $\chi(\mathbf{x}, M) = \chi(\mathbf{x}, x_1 M)$. Therefore, $\chi(\mathbf{x}, M) = \chi(\mathbf{x}, x_1^j M)$ for all $j \geq 1$. Consequently, $\chi(\mathbf{x}, M) = \chi(\mathbf{x}, x_1^u M) = \chi(\mathbf{x}, 0_M) = 0$. We get (ii).

The proof of (iii): Since x_1 is M-regular, we have

$$H_i(\mathbf{x}, M) \cong H_i(\mathbf{x}', M/x_1M)$$

by [3, Corollary 1.6.13(b)]. Thus

$$\ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}] = \ell_A[H_i(\mathbf{x}', M/x_1M)_{\mathbf{n}}]$$

for all large $\mathbf{n}$ and for all i. From this it follows (iii). $\square$

Using Lemma 3.5, we prove the following recursion formula for the Euler-Poincare characteristic of a graded module with respect to a mixed multiplicity system.

Lemma 3.6. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{x} = x_1, \ldots, x_n$ be a mixed multiplicity system of M. Set $\mathbf{x}' = x_2, \ldots, x_n$. Then we have*

$$\chi(\mathbf{x}, M) = \chi(\mathbf{x}', M/x_1M) - \chi(\mathbf{x}', 0_M : x_1).$$

Proof. Consider the following cases.

If $x_1 \in \sqrt{\mathrm{Ann}_S M}$ then $\chi(\mathbf{x}, M) = 0$ by Lemma 3.5(ii). In this case,

$$\mathrm{Supp} M/\mathbf{x}M = \mathrm{Supp} M/\mathbf{x}'M.$$

Hence $\mathbf{x}'$ is also a mixed multiplicity system of M. Hence since the exact sequence

$$0 \longrightarrow (0_M : x_1) \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

we have $\chi(\mathbf{x}', M/x_1M) - \chi(\mathbf{x}', 0_M : x_1) = 0$ by Lemma 3.5(i). So

$$\chi(\mathbf{x}, M) = \chi(\mathbf{x}', M/x_1M) - \chi(\mathbf{x}', 0_M : x_1).$$

In the case that $x_1 \notin \sqrt{\mathrm{Ann}_S M}$ and set $D = 0_M : x_1^\infty$, x_1 is regular on M/D. Then since $x_1 M \bigcap D = x_1 D$, we have the exact sequence

$$0 \longrightarrow \frac{D}{x_1 D} \longrightarrow \frac{M}{x_1 M} \longrightarrow \frac{M}{x_1 M + D} \longrightarrow 0.$$

Hence
$$\chi(\mathbf{x}', \frac{M}{x_1M + D}) = \chi(\mathbf{x}', \frac{M}{x_1M}) - \chi(\mathbf{x}', \frac{D}{x_1D})$$
by Lemma 3.5(i). Remember that M is noetherian, $D = 0_M : x_1^v$ for some $v > 0$. Hence $x_1 \in \sqrt{\mathrm{Ann}_S D}$. So $\mathbf{x}'$ is also a mixed multiplicity system of D. On the other hand $\chi(\mathbf{x}', \frac{D}{x_1D}) = \chi(\mathbf{x}', 0_M : x_1)$ since the exact sequence
$$0 \longrightarrow (0_M : x_1) \longrightarrow D \xrightarrow{x_1} D \longrightarrow \frac{D}{x_1D} \longrightarrow 0.$$
Thus
$$\chi(\mathbf{x}', \frac{M}{x_1M + D}) = \chi(\mathbf{x}', \frac{M}{x_1M}) - \chi(\mathbf{x}', 0_M : x_1). \tag{3.3}$$
Recall that x_1 is regular on M/D,
$$\chi(\mathbf{x}', \frac{M}{x_1M + D}) = \chi(\mathbf{x}, \frac{M}{D})$$
by Lemma 3.5(iii). Now since
$$\chi(\mathbf{x}, \frac{M}{D}) = \chi(\mathbf{x}, M) - \chi(\mathbf{x}, D)$$
by Lemma 3.5(i), we get
$$\chi(\mathbf{x}', \frac{M}{x_1M + D}) = \chi(\mathbf{x}, M) - \chi(\mathbf{x}, D).$$
Remember that $x_1 \in \sqrt{\mathrm{Ann}_S D}$, $\chi(\mathbf{x}, D) = 0$ by Lemma 3.5(ii). So
$$\chi(\mathbf{x}', \frac{M}{x_1M + D}) = \chi(\mathbf{x}, M). \tag{3.4}$$
By (3.3) and (3.4) we obtain
$$\chi(\mathbf{x}, M) = \chi(\mathbf{x}', M/x_1M) - \chi(\mathbf{x}', 0_M : x_1).$$
The lemma is proved. $\square$

Next we construct an invariant that is called the mixed multiplicity symbol with respect to a mixed multiplicity system.

Definition 3.7. Let $\mathbf{x} = x_1, \ldots, x_n$ be a mixed multiplicity system of M. If $n = 0$, then $\ell_A[M_{\mathbf{n}}] = c$ (const) for all $\mathbf{n} \gg 0$ and we set $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\emptyset, M) = c$. If $n > 0$, we set
$$\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/x_1M) - \widetilde{e}(\mathbf{x}', 0_M : x_1),$$
here $\mathbf{x}' = x_2, \ldots, x_n$. We call $\widetilde{e}(\mathbf{x}, M)$ the *mixed multiplicity symbol of M with respect to* $\mathbf{x}$.

Then the relationship between the Euler-Poincare characteristic and the mixed multiplicity symbol with respect to a mixed multiplicity system of M is given by the following proposition.

Proposition 3.8. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{x}$ be a mixed multiplicity system of M. Then we have*

$$\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M).$$

Proof. Note that if the length of $\mathbf{x}$ is equal to 0, then we have

$$\chi(\mathbf{x}, M) = \ell_A[H_0(\mathbf{x}, M)_{\mathbf{n}}] = \ell_A(M_{\mathbf{n}}) = \widetilde{e}(\mathbf{x}, M)$$

for all large $\mathbf{n}$. Consequently the assertion follows from Lemma 3.6 and the definition of the mixed multiplicity symbol of M with respect to $\mathbf{x}$ (see Definition 3.7). $\square$

The notion of mixed multiplicities $e(M; \mathbf{k})$ of the type $\mathbf{k}$ of a module M always requires the condition

$$\mid \mathbf{k} \mid = \dim \operatorname{Supp}_{++} M.$$

This sometimes becomes a obstruction in describing the relationship between mixed multiplicities and the Euler-Poincare characteristic and in expressing mixed multiplicity formulas. Consequently, we need the following extension.

Definition 3.9. For each $\mathbf{k} \in \mathbb{N}^d$ such that $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M$, we put

$$E(M; \mathbf{k}) = \begin{cases} e(M; \mathbf{k}) & \text{if } \mid \mathbf{k} \mid = \dim \operatorname{Supp}_{++} M, \\ \\ 0 & \text{if } \mid \mathbf{k} \mid > \dim \operatorname{Supp}_{++} M. \end{cases}$$

Remember that for any $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M$, there exists a mixed multiplicity system of M of the type $\mathbf{k}$ by Remark 3.3(i). Then the mixed multiplicity of M of the type $\mathbf{k}$ and the Euler-Poincare characteristic of a mixed multiplicity system of M of the type $\mathbf{k}$ are directly linked by the following proposition.

Proposition 3.10. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{x}$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Assume that $\dim \operatorname{Supp}_{++} M \leq \mid \mathbf{k} \mid$. Then we have*

$$E(M; \mathbf{k}) = \chi(\mathbf{x}, M).$$

Proof. Let $P_M(\mathbf{n})$ be the Hilbert polynomial of the Hilbert function $\ell_A[M_{\mathbf{n}}]$. Set $\mid \mathbf{k} \mid = s$. Then we have $\deg P_M(\mathbf{n}) \leq s$. Denote by r the number of all non-zero elements in $\mathbf{x}$. We first will prove that

$$\chi(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n})$$

by induction on r.

Consider the case that $r = 0$. Then $\dim \mathrm{Supp}_{++} M \leq 0$, so $\deg P_M(\mathbf{n}) \leq 0$. Now if $s = 0$, then we have

$$\chi(\mathbf{x}, M) = \ell_A[H_0(\mathbf{x}, M)_{\mathbf{n}}] = \ell_A[M_{\mathbf{n}}]$$

for all $\mathbf{n} \gg 0$ and

$$\triangle^{\mathbf{k}} P_M(\mathbf{n}) = \triangle^{\mathbf{0}} P_M(\mathbf{n}) = \ell_A[M_{\mathbf{n}}]$$

for all $\mathbf{n} \gg 0$. Hence

$$\chi(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n}).$$

If $s > 0$, then $\chi(\mathbf{x}, M) = 0$ by Lemma 3.5(ii). Note that $\mid \mathbf{k} \mid = s > 0$, $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = 0$ since $\deg P_M(\mathbf{n}) \leq 0$. Thus

$$\chi(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n}).$$

Therefore, if $r = 0$ then $\chi(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n})$. Consequently the result is true for $r = 0$.

Next assume that $r > 0$; $\mathbf{x} = x_1, \ldots, x_s$ and $x_1 \neq 0$. Set $\mathbf{x}' = x_2, \ldots, x_s$. Let

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_u = M$$

be a prime filtration of M, i.e., $M_{j+1}/M_j \cong S/P_j$ where P_j is a homogeneous prime ideal for all $0 \leq j \leq u-1$ and

$$\{P_0, P_1, \ldots, P_{u-1}\} \subseteq \mathrm{Supp} M.$$

Then we get $\dim \mathrm{Supp}_{++} S/P_j \leq \dim \mathrm{Supp}_{++} M \leq \mid \mathbf{k} \mid$ for all $0 \leq j \leq u-1$ and

$$P_M(\mathbf{n}) = \sum_{j=0}^{u-1} P_{S/P_j}(\mathbf{n}).$$

By Lemma 3.4, $\mathbf{x}$ is a mixed multiplicity system of S/P_j for each $0 \leq j \leq u-1$. And by Lemma 3.5(i), it follows that

$$\chi(\mathbf{x}, M) = \sum_{j=0}^{u-1} \chi(\mathbf{x}, S/P_j).$$

Now if $x_1 \in P_j$ and denote by $\overline{\mathbf{x}}$ the image of $\mathbf{x}$ in S/P_j and consider $\chi(\overline{\mathbf{x}}, S/P_j)$ as the Euler-Poincare characteristic of the S/P_j-module S/P_j with respect to $\overline{\mathbf{x}}$ then

$$\chi(\overline{\mathbf{x}}, S/P_j) = \chi(\mathbf{x}, S/P_j)$$

and $\overline{x}_1 = 0$ in S/P_j. Hence if we call v the number of all non-zero elements in $\overline{\mathbf{x}}$ then $v < r$. Note that we can also consider $P_{S/P_j}(\mathbf{n})$ as the Hilbert polynomial of the Hilbert function $\ell_A[(S/P_j)_{\mathbf{n}}]$ in the S/P_j-module S/P_j. Then since $\dim \operatorname{Supp}_{++} S/P_j \leq |\ \mathbf{k}\ |$, by the inductive assumption we have

$$\triangle^{\mathbf{k}} P_{S/P_j}(\mathbf{n}) = \chi(\overline{\mathbf{x}}, S/P_j).$$

So

$$\triangle^{\mathbf{k}} P_{S/P_j}(\mathbf{n}) = \chi(\mathbf{x}, S/P_j).$$

In the case that $x_1 \notin P_j$ then x_1 is S/P_j-regular of S-module S/P_j. Now assume that $x_1 \in S_i$. Then $k_i > 0$; and by Remark 2.5 (see [31, Remark 2.6]), it follows that

$$\triangle^{\mathbf{e}_i} P_{S/P_j}(\mathbf{n}) = P_{S/(x_1,P_j)}(\mathbf{n}) \ \text{ and}$$
$$\dim \operatorname{Supp}_{++} S/(x_1, P_j) \leq \dim \operatorname{Supp}_{++} S/P_j - 1.$$

Remember that

$$\chi(\mathbf{x}', S/(x_1, P_j)) = \chi(\mathbf{x}, S/P_j)$$

by Lemma 3.5(iii). But the number of all non-zero elements in $\mathbf{x}'$ is less than r and $\mathbf{x}'$ is a mixed multiplicity system of the type $\mathbf{k} - \mathbf{e}_i \in \mathbb{N}^d$ of $S/(x_1, P_j)$; and

$$\begin{aligned}\dim \operatorname{Supp}_{++} S/(x_1, P_j) &\leq \dim \operatorname{Supp}_{++} S/P_j - 1 \\ &\leq \dim \operatorname{Supp}_{++} M - 1 \leq |\ \mathbf{k} - \mathbf{e}_i\ |,\end{aligned}$$

by the inductive assumption, we get

$$\triangle^{\mathbf{k}-\mathbf{e}_i} P_{S/(x_1,P_j)}(\mathbf{n}) = \chi(\mathbf{x}', S/(x_1, P_j)).$$

Note that

$$\triangle^{\mathbf{k}-\mathbf{e}_i} P_{S/(x_1,P_j)}(\mathbf{n}) = \triangle^{\mathbf{k}-\mathbf{e}_i}[\triangle^{\mathbf{e}_i} P_{S/P_j}(\mathbf{n})] = \triangle^{\mathbf{k}} P_{S/P_j}(\mathbf{n}).$$

So

$$\triangle^{\mathbf{k}} P_{S/P_j}(\mathbf{n}) = \chi(\mathbf{x}, S/P_j).$$

Therefore, for each $0 \leq j \leq u - 1$, we always obtain

$$\triangle^{\mathbf{k}} P_{S/P_j}(\mathbf{n}) = \chi(\mathbf{x}, S/P_j).$$

Recall that $P_M(\mathbf{n}) = \sum_{j=0}^{u-1} P_{S/P_j}(\mathbf{n})$ and $\chi(\mathbf{x}, M) = \sum_{j=0}^{u-1} \chi(\mathbf{x}, S/P_j)$. Consequently

$$\triangle^{\mathbf{k}} P_M(\mathbf{n}) = \sum_{j=0}^{u-1} \triangle^{\mathbf{k}} P_{S/P_j}(\mathbf{n}) = \sum_{j=0}^{u-1} \chi(\mathbf{x}, S/P_j) = \chi(\mathbf{x}, M).$$

Thus $\chi(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n})$. Induction is complete. Hence the equation

$$\chi(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n})$$

is proved. Now remember that $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = e(M; \mathbf{k})$ if $|\ \mathbf{k}\ | = \dim \operatorname{Supp}_{++} M$, and $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = 0$ if $|\ \mathbf{k}\ | > \dim \operatorname{Supp}_{++} M$. Thus $\chi(\mathbf{x}, M) = E(M; \mathbf{k})$. $\square$

The proof of Proposition 3.10 yields:

Remark 3.11. We have $\chi(\mathbf{x}, M) = E(M; \mathbf{k}) = \triangle^{\mathbf{k}} P_M(\mathbf{n})$. Hence if

$$|\ \mathbf{k}\ | > \dim \operatorname{Supp}_{++} M$$

then $\chi(\mathbf{x}, M) = 0$ since $E(M; \mathbf{k}) = 0$. So $\chi(\mathbf{x}, M) \neq 0$ and hence $\widetilde{e}(\mathbf{x}, M) \neq 0$ by Proposition 3.8 if and only if $|\ \mathbf{k}\ | = \dim \operatorname{Supp}_{++} M$ and $e(M; \mathbf{k}) = E(M; \mathbf{k}) \neq 0$. In particular, if $\dim \operatorname{Supp}_{++} M < 0$ then $E(M; \mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{N}^d$ and $\mathbf{x}$ is a mixed multiplicity system of M for each finite sequence $\mathbf{x} \subseteq \bigcup_{j=1}^{d} S_j$. Therefore,

$$\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = 0$$

for each finite sequence $\mathbf{x} \subseteq \bigcup_{j=1}^{d} S_j$.

Combining Proposition 3.8 with Proposition 3.10, we obtain the following main result of this paper.

Theorem 3.12. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{x}$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Assume that $\dim \operatorname{Supp}_{++} M \leq |\ \mathbf{k}\ |$. Then we have*

$$E(M; \mathbf{k}) = \chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M).$$

Now assume that $\mathbf{x}$ is a mixed multiplicity system of M and $a \in \mathbf{x}$ is an S_{++}-filter-regular element with respect to M. Set $\mathbf{x}' = \mathbf{x} \backslash \{a\}$. Since a is an S_{++}-filter-regular element, $P_{(0_M : a)}(\mathbf{n}) = 0$ and hence $\dim \operatorname{Supp}_{++}(0_M : a) < 0$. Consequently $\widetilde{e}(\mathbf{x}', 0_M : a) = 0$ by Remark 3.11. Therefore

$$\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/aM) - \widetilde{e}(\mathbf{x}', 0_M : a) = \widetilde{e}(\mathbf{x}', M/aM).$$

From this it follows that $\chi(\mathbf{x}, M) = \chi(\mathbf{x}', M/aM)$ by Proposition 3.8.

We get the following corollary.

Corollary 3.13. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{x}$ be a mixed multiplicity system of M. Let $a \in \mathbf{x}$ be an S_{++}-filter-regular element with respect to M. Set $\mathbf{x}' = \mathbf{x} \setminus \{a\}$. Then*

(i) $\chi(\mathbf{x}, M) = \chi(\mathbf{x}', M/aM)$.

(ii) $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/aM)$.

Next, we will build the recursion formula for mixed multiplicities that is compatible with the recursion formula for Euler-Poincare characteristics.

We need the following comments.

Remark 3.14. Let $a \in S_i$. Then we have $\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n}) - P_{(0_M:\, a)}(\mathbf{n} - \mathbf{e}_i)$ by Proposition 2.6. Since $\deg \triangle^{\mathbf{e}_i} P_M(\mathbf{n}) \leq \deg P_M(\mathbf{n}) - 1 = \dim \operatorname{Supp}_{++} M - 1$, it follows that $\deg P_{M/aM}(\mathbf{n}) \leq \dim \operatorname{Supp}_{++} M - 1$ if and only if

$$\deg P_{(0_M:\, a)}(\mathbf{n}) \leq \dim \operatorname{Supp}_{++} M - 1.$$

Hence $\dim \operatorname{Supp}_{++} \left(\dfrac{M}{aM} \right) \leq \dim \operatorname{Supp}_{++} M - 1$ if and only if

$$\dim \operatorname{Supp}_{++} (0_M : a) \leq \dim \operatorname{Supp}_{++} M - 1.$$

Let $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $k_i > 0$ and $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M$. Let $a \in S_i$. Assume that $\dim \operatorname{Supp}_{++} \left(\dfrac{M}{aM} \right) \leq \dim \operatorname{Supp}_{++} M - 1$. Then it follows by Remark 3.14 that $\dim \operatorname{Supp}_{++} (0_M : a) \leq \dim \operatorname{Supp}_{++} M - 1$. It can be verified that there exists a mixed multiplicity system $\mathbf{x}$ of M of the type $\mathbf{k}$ such that $a \in \mathbf{x}$ by Remark 3.3(i). Set $\mathbf{x}' = \mathbf{x} \setminus \{a\}$. It is clear that $\mathbf{x}'$ is a mixed multiplicity system of both M/aM and $(0_M : a)$ of the type $\mathbf{k} - \mathbf{e}_i$. By Theorem 3.12, we get $E(M; \mathbf{k}) = \chi(\mathbf{x}, M)$ and

$$E(M/aM; \mathbf{k} - \mathbf{e}_i) = \chi(\mathbf{x}', M/aM);$$
$$E(0_M : a; \mathbf{k} - \mathbf{e}_i) = \chi(\mathbf{x}', 0_M : a).$$

Hence $E(M; \mathbf{k}) = E(M/aM; \mathbf{k} - \mathbf{e}_i) - E(0_M : a; \mathbf{k} - \mathbf{e}_i)$ by Lemma 3.6.

In particular, if a is an S_{++}-filter-regular element of M, then $P_{(0_M:a)}(\mathbf{n}) = 0$. Hence $E(0_M : a; \mathbf{k} - \mathbf{e}_i) = 0$. Moreover $\dim \operatorname{Supp}_{++} \left(\dfrac{M}{aM} \right) \leq \dim \operatorname{Supp}_{++} M - 1$ by Remark 2.5. In this case, we get $E(M; \mathbf{k}) = E(M/aM; \mathbf{k} - \mathbf{e}_i)$.

The above facts yield:

Corollary 3.15. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $a \in S_i$. Assume that $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ satisfies $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M;\ k_i > 0$. Then the following statements hold.*

(i) *If* $\dim \operatorname{Supp}_{++}\left(\dfrac{M}{aM}\right) \leq \dim \operatorname{Supp}_{++} M - 1$ *then*

$$E(M; \mathbf{k}) = E(M/aM; \mathbf{k} - \mathbf{e}_i) - E(0_M : a; \mathbf{k} - \mathbf{e}_i).$$

(ii) *If a is an S_{++}-filter-regular element of M then* $E(M; \mathbf{k}) = E(M/aM; \mathbf{k} - \mathbf{e}_i)$.

Note that Corollary 3.15(ii) is a generalization of [31, Proposition 3.3(i)].

Recall that $S_{\mathbf{1}} = S_{(1,\ldots,1)}$. Now if $S_{\mathbf{1}} \nsubseteq \sqrt{\operatorname{Ann}_S M}$ then $\dim \operatorname{Supp}_{++} M \geq 0$. Set $\dim \operatorname{Supp}_{++} M = s$. Then we have $E(M; \mathbf{k}) = e(M; \mathbf{k})$ for each $\mathbf{k} \in \mathbb{N}^d$ with $\mid \mathbf{k} \mid = s$. Hence from Theorem 3.12 we immediately get the following strong result.

Theorem 3.16. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module such that $S_{\mathbf{1}}$ is not contained in $\sqrt{\operatorname{Ann}_S M}$. Set $\dim \operatorname{Supp}_{++} M = s$. Assume that $\mathbf{x}$ is a mixed multiplicity system of M of the type $\mathbf{k} \in \mathbb{N}^d$ with $\mid \mathbf{k} \mid = s$. Then we have*

$$e(M; \mathbf{k}) = \chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M).$$

Remark 3.17. Theorem 3.16 proves that the mixed multiplicity symbol of M with respect to a mixed multiplicity system $\mathbf{x}$ does not depend on the order of the elements of $\mathbf{x}$ and the Euler-Poincare characteristic $\chi(\mathbf{x}, M)$ depends only on the type of $\mathbf{x}$.

Remember that for any $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ with $\mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++} M$, there exists an S_{++}-filter-regular sequence $\mathbf{x} = x_1, \ldots, x_s$ of M of the type $\mathbf{k}$. Then $\mathbf{x}$ is also a mixed multiplicity system of M by Remark 3.3(i). And we have

$$\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(x_{i+1}, \ldots, x_s, M/(x_1, \ldots, x_i)M)$$

for each $1 \leq i \leq s$ by Corollary 3.13(ii). Consequently, $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}\left(\emptyset, \frac{M}{\mathbf{x}M}\right)$. Note that $E\left(\frac{M}{\mathbf{x}M}; \mathbf{0}\right) = \widetilde{e}\left(\emptyset, \frac{M}{\mathbf{x}M}\right)$ by Theorem 3.12. Therefore

$$\widetilde{e}(\mathbf{x}, M) = \ell_A\left(\frac{M}{\mathbf{x}M}\right)_{\mathbf{n}} = E\left(\frac{M}{\mathbf{x}M}; \mathbf{0}\right)$$

for all $\mathbf{n} \gg 0$. So $\ell_A\big(\frac{M}{\mathbf{x}M}\big)_{\mathbf{n}} \neq 0$ for all $\mathbf{n} \gg 0$ and hence $\widetilde{e}(\mathbf{x}, M) \neq 0$ if and only if $\dim \operatorname{Supp}_{++}\big(\frac{M}{\mathbf{x}M}\big) = 0$. And in this case, $|\, \mathbf{k} \,| = \dim \operatorname{Supp}_{++} M$ by Remark 3.11.

Hence we have the following result that combines Theorem 2.4 ([31, Theorem 3.4]) with Theorem 3.12.

Corollary 3.18. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module. Let $\mathbf{k} \in \mathbb{N}^d$ with $|\, \mathbf{k} \,| \geq \dim \operatorname{Supp}_{++} M$. Assume that $\mathbf{x}$ is an S_{++}-filter-regular sequence with respect to M of the type $\mathbf{k}$. Then we have*

$$E(M; \mathbf{k}) = \chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = E\left(\frac{M}{\mathbf{x}M}; \mathbf{0}\right) = \ell_A\left(\frac{M}{\mathbf{x}M}\right)_{\mathbf{n}}$$

for all large $\mathbf{n}$. And $E(M; \mathbf{k}) \neq 0$ if and only if $\dim \operatorname{Supp}_{++}\big(\frac{M}{\mathbf{x}M}\big) = 0$. In this case, $|\, \mathbf{k} \,| = \dim \operatorname{Supp}_{++} M$.

For any $\mathbf{k} \in \mathbb{N}^d$ with $|\, \mathbf{k} \,| = \dim \operatorname{Supp}_{++} M$, there exists a mixed multiplicity system $\mathbf{x} = x_1, \ldots, x_s$ of M of the type $\mathbf{k}$ by Remark 3.3(i). By Theorem 3.16, $e(M; \mathbf{k}) = \widetilde{e}(\mathbf{x}, M)$. Set $\mathbf{x}' = x_2, \ldots, x_s$. And assume that $x_1 \in S_i$. Then $k_i > 0$. Since

$$\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/x_1M) - \widetilde{e}(\mathbf{x}', 0_M : x_1),$$

it follows that $\widetilde{e}(\mathbf{x}', M/x_1M) \geq e(M; \mathbf{k})$. Note that $\mathbf{x}'$ is a mixed multiplicity system of the type $\mathbf{k} - \mathbf{e}_i$ of both M/x_1M and $0_M : x_1$. Now assume that $e(M; \mathbf{k}) > 0$. Then we have $\widetilde{e}(\mathbf{x}', M/x_1M) > 0$. Therefore $s - 1 = \dim \operatorname{Supp}_{++}\left[\frac{M}{x_1M}\right]$ by Remark 3.11. So in this case,

$$\dim \operatorname{Supp}_{++}\left[\frac{M}{x_1M}\right] = \dim \operatorname{Supp}_{++} M - 1.$$

Hence $\dim \operatorname{Supp}_{++}(0_M : x_1) \leq \dim \operatorname{Supp}_{++} M - 1$ by Remark 3.14. Remember that by Theorem 3.16, we have $e(M/x_1M; \mathbf{k} - \mathbf{e}_i) = \widetilde{e}(\mathbf{x}', M/x_1M)$. Therefore,

$$e(M/x_1M; \mathbf{k} - \mathbf{e}_i) > 0.$$

Consequently, by induction we easily get

$$\dim \operatorname{Supp}_{++}\left[\frac{M}{(x_1, \ldots, x_i)M}\right] = \dim \operatorname{Supp}_{++} M - i \geq \dim \operatorname{Supp}_{++} \frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}$$

for all $1 \leq i \leq s$. In this case, $\dim \operatorname{Supp}_{++}\big[\frac{M}{\mathbf{x}M}\big] = 0$ and $\widetilde{e}\big(\emptyset, \frac{M}{\mathbf{x}M}\big) = \ell_A\big(\frac{M}{\mathbf{x}M}\big)_{\mathbf{n}}$ for all large $\mathbf{n}$. Denote by $\mathbf{h}_i = (h_{i1}, \ldots, h_{id})$ the type of a subsequence $x_1, \ldots, x_i$ of $\mathbf{x}$ for each $1 \leq i \leq s$. Then it is easily seen that $x_{i+1}, \ldots, x_s$ is a mixed multiplicity system of $\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}$ of the type $\mathbf{k} - \mathbf{h}_i$ and

$$\dim \operatorname{Supp}_{++} \frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M} \leq |\, \mathbf{k} - \mathbf{h}_i \,|$$

for all $1 \leq i \leq s$. Hence

$$\widetilde{e}\left(x_{i+1}, \ldots, x_s, \frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}\right) = E\left(\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}; \mathbf{k} - \mathbf{h}_i\right)$$

for all $1 \leq i \leq s$ by Theorem 3.12. On the one hand

$$\widetilde{e}(\mathbf{x}, M) = \widetilde{e}\left(\emptyset, \frac{M}{\mathbf{x}M}\right) - \sum_{i=1}^{s} \widetilde{e}\left(x_{i+1}, \ldots, x_s, \frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}\right)$$

by Definition 3.7. On the other hand $e(M; \mathbf{k}) = \widetilde{e}(\mathbf{x}, M)$ and $e\big(\frac{M}{\mathbf{x}M}; \mathbf{0}\big) = \widetilde{e}\big(\emptyset, \frac{M}{\mathbf{x}M}\big)$ by Theorem 3.16. Thus, we have $e(M; \mathbf{k}) = e\left(\frac{M}{\mathbf{x}M}; \mathbf{0}\right) - \sum_{i=1}^{s} E\left(\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}; \mathbf{k} - \mathbf{h}_i\right)$.

The above remarks yield:

Corollary 3.19. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module such that $S_{\mathbf{1}}$ is not contained in $\sqrt{\mathrm{Ann}_S M}$. Let $\mathbf{x} = x_1, \ldots, x_s$ be a mixed multiplicity system of M of the type $\mathbf{k}$ with $|\, \mathbf{k} \,| = \dim \mathrm{Supp}_{++} M$. Denote by $\mathbf{h}_i = (h_{i1}, \ldots, h_{id})$ the type of a subsequence $x_1, \ldots, x_i$ of $\mathbf{x}$ for each $1 \leq i \leq s$. Assume that $e(M; \mathbf{k}) \neq 0$. Then the following statements hold.*

(i) *For each $1 \leq i \leq s$, we have* $\dim \mathrm{Supp}_{++}\left[\frac{M}{(x_1, \ldots, x_i)M}\right] = \dim \mathrm{Supp}_{++} M - i$.

(ii) $e\big(\frac{M}{\mathbf{x}M}; \mathbf{0}\big) = \ell_A\big(\frac{M}{\mathbf{x}M}\big)_{\mathbf{n}}$ *for all large* $\mathbf{n}$ *and*

$$e(M; \mathbf{k}) = e\left(\frac{M}{\mathbf{x}M}; \mathbf{0}\right) - \sum_{i=1}^{s} E\left(\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}; \mathbf{k} - \mathbf{h}_i\right).$$

Let Λ be the set of all homogeneous prime ideals P of S such that

$$P \in \mathrm{Supp}_{++} M \text{ and } \dim \mathrm{Proj}\ (S/P) = \dim \mathrm{Supp}_{++} M.$$

Then by [33, Theorem 3.1], we have $e(M; \mathbf{k}) = \sum_{P \in \Lambda} \ell(M_P) e(S/P; \mathbf{k})$. Recall that if $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_u = M$ is a prime filtration of M, then for any $P \in \Lambda$, there exists $0 \leq i \leq u - 1$ such that $S/P \cong M_{i+1}/M_i$ by the proof of [33, Theorem 3.1]. Therefore, if $\mathbf{x}$ is a mixed multiplicity system of M then $\mathbf{x}$ is also a mixed multiplicity system of S/P for any $P \in \Lambda$ by Lemma 3.4.

Hence by Theorem 3.16, we immediately obtain the following corollary.

Corollary 3.20. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an artinian local ring A and let M be a finitely generated $\mathbb{N}^d$-graded S-module such that $S_{\mathbf{1}}$ is not contained in $\sqrt{\mathrm{Ann}_S M}$. Denote by Λ the set of all homogeneous prime ideals P of S such that $P \in \mathrm{Supp}_{++}M$ and $\dim \mathrm{Proj}(S/P) = \dim \mathrm{Supp}_{++}M$. Set $\dim \mathrm{Supp}_{++}M = s$. Assume that $\mathbf{x}$ is a mixed multiplicity system of M of the type $\mathbf{k}$ with $|\,\mathbf{k}\,| = s$. Then*

(i) $\chi(\mathbf{x}, M) = \sum_{P\in\Lambda} \ell(M_P)\chi(\mathbf{x}, S/P)$.

(ii) $\widetilde{e}(\mathbf{x}, M) = \sum_{P\in\Lambda} \ell(M_P)\widetilde{e}(\mathbf{x}, S/P)$.

4 Applications to Mixed Multiplicities of Ideals

In this section, we will give some applications of Section 3 to mixed multiplicities of modules over local rings with respect to ideals.

Let $(R, \mathfrak{n})$ be a noetherian local ring with maximal ideal $\mathfrak{n}$ and infinite residue field $R/\mathfrak{n}$. Let N be a finitely generated R-module. Let $J, I_1, \ldots, I_d$ be ideals of R with J being $\mathfrak{n}$-primary and $I_1 \cdots I_d \nsubseteq \sqrt{\mathrm{Ann}_R N}$. Recall that $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$; $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$; $\mathbf{I} = I_1, \ldots, I_d$; $\mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \ldots, I_d^{[k_d]}$ and $\mathbb{I}^{\mathbf{n}} = I_1^{n_1} \cdots I_d^{n_d}$. We get an $\mathbb{N}^{(d+1)}$-graded algebra and an $\mathbb{N}^{(d+1)}$-graded module:

$$T = \bigoplus_{n\geq 0,\ \mathbf{n}\geq \mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}}{J^{n+1}\mathbb{I}^{\mathbf{n}}} \text{ and } \mathcal{N} = \bigoplus_{n\geq 0,\ \mathbf{n}\geq \mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}N}{J^{n+1}\mathbb{I}^{\mathbf{n}}N}.$$

Then T is a finitely generated standard $\mathbb{N}^{(d+1)}$-graded algebra over an artinian local ring R/J and $\mathcal{N}$ is a finitely generated $\mathbb{N}^{(d+1)}$-graded T-module. The mixed multiplicity of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ is denoted by $e\big(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N\big)$, i.e.,

$$e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_d^{[k_d]}; N) := e(\mathcal{N}; k_0, k_1, \ldots, k_d)$$

and which is called the *mixed multiplicity of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0 + 1, \mathbf{k})$ (see [8, 14, 24]). Set $I = JI_1 \cdots I_d$ and $I_0 = J$.

Remark 4.1. It can be verified that $I \subseteq \sqrt{\mathrm{Ann}_R N}$ if and only if $T_{(1,\mathbf{1})} \subseteq \sqrt{\mathrm{Ann}_T \mathcal{N}}$. In this case, $\mathrm{Supp}_{++}\mathcal{N} = \emptyset$ and $\dfrac{N}{0_N : I^\infty} = 0$. Then we assign

$$\dim \mathrm{Supp}_{++}\mathcal{N} = \dim \frac{N}{0_N : I^\infty} = -\infty.$$

If $I \nsubseteq \sqrt{\mathrm{Ann}_R N}$ then on the one hand

$$\deg P_{\mathcal{N}}(\mathbf{n}) = \dim \mathrm{Supp}_{++}\mathcal{N}$$

by [8, Theorem 4.1], and on the other hand

$$\deg P_{\mathcal{N}}(\mathbf{n}) = \dim \frac{N}{0_N : I^\infty} - 1$$

by [25, Proposition 3.1(i)](see [14]). Hence

$$\dim \mathrm{Supp}_{++}\mathcal{N} = \dim \frac{N}{0_N : I^\infty} - 1.$$

So we always have $\dim \mathrm{Supp}_{++}\mathcal{N} = \dim \dfrac{N}{0_N : I^\infty} - 1$ in all cases.

Note that if $a \in I_i$ and a^* is the image of a in $T_i = T_{(0,\ldots,\underset{(i)}{1},\ldots,0)}$ for $0 \leq i \leq d$ then

$$[\mathcal{N}/a^*\mathcal{N}]_{(m,\,\mathbf{m})} \cong \Big[\bigoplus_{n\geq 0,\,\mathbf{n}\geq \mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}(N/aN)}{J^{n+1}\mathbb{I}^{\mathbf{n}}(N/aN)}\Big]_{(m,\,\mathbf{m})}$$

for all $m \gg 0$; $\mathbf{m} \gg \mathbf{0}$ if and only if

$$aN \bigcap J^m\mathbb{I}^{\mathbf{m}}I_iN \equiv aJ^m\mathbb{I}^{\mathbf{m}}N \pmod{J^{m+1}\mathbb{I}^{\mathbf{m}}I_iN}$$

for all $m \gg 0$; $\mathbf{m} \gg \mathbf{0}$ by [31, Remark 4.1]. This comment is the reason for using the following sequence.

Definition 4.2. An element $a \in R$ is called a *Rees's superficial element* of N with respect to $\mathbf{I}$ if there exists $i \in \{1, \ldots, d\}$ such that $a \in I_i$ and

$$aN \bigcap \mathbb{I}^{\mathbf{n}}I_iN = a\mathbb{I}^{\mathbf{n}}N$$

for all $\mathbf{n} \gg 0$. Let $x_1, \ldots, x_t$ be a sequence in R. Then $x_1, \ldots, x_t$ is called a *Rees's superficial sequence* of N with respect to $\mathbf{I}$ if x_{j+1} is a Rees's superficial element of $N/(x_1, \ldots, x_j)N$ with respect to $\mathbf{I}$ for all $j = 0, 1, \ldots, t-1$. A Rees's superficial sequence of N consisting of k_1 elements of $I_1, \ldots, k_d$ elements of I_d is called a Rees's superficial sequence of N of the type $\mathbf{k} = (k_1, \ldots, k_d)$.

Remark 4.3. It can be verified that if $I_1 \cdots I_d$ is contained in $\sqrt{\mathrm{Ann}_R N}$ then a is a Rees's superficial element of N with respect to $\mathbf{I}$ for all $a \in \bigcup_{i=1}^d I_i$. Consequently, by [14, Lemma 2.2] which is a generalized version of [15, Lemma 1.2], for any set of ideals $I_1, \ldots, I_d$ and each $(k_1, \ldots, k_d) \in \mathbb{N}^d$, there exists a Rees's superficial sequence in $\bigcup_{i=1}^d I_i$ with respect to M consisting of k_1 elements of $I_1, \ldots, k_d$ elements of I_d.

Let $\mathbf{x}$ be a Rees's superficial sequence of N with respect to ideals $J, \mathbf{I}$ and let $\mathbf{x}^*$ be the image of $\mathbf{x}$ in $\bigcup_{i=0}^{d} T_i$. It can be verified (or see [31, Remark 4.1]) that

$$[\mathcal{N}/\mathbf{x}^*\mathcal{N}]_{(m,\,\mathbf{m})} \cong \Big[\bigoplus_{n\geq 0,\,\mathbf{n}\geq \mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}(N/\mathbf{x}N)}{J^{n+1}\mathbb{I}^{\mathbf{n}}(N/\mathbf{x}N)}\Big]_{(m,\,\mathbf{m})} \tag{4.1}$$

for all $m \gg 0;\ \mathbf{m} \gg \mathbf{0}$. Hence by Remark 4.1, we have

$$\dim \operatorname{Supp}_{++}\frac{\mathcal{N}}{\mathbf{x}^*\mathcal{N}} = \dim \frac{N}{\mathbf{x}N : I^{\infty}} - 1.$$

So $\mathbf{x}^*$ is a mixed multiplicity system of $\mathcal{N}$ if and only if $\dim \dfrac{N}{\mathbf{x}N : I^{\infty}} \leq 1$.

These comments help us define the following system.

Definition 4.4. Let $\mathbf{x}$ be a Rees's superficial sequence of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k}) \in \mathbb{N}^{d+1}$. Then $\mathbf{x}$ is called a *mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0, \mathbf{k})$ if $\dim \dfrac{N}{\mathbf{x}N : I^{\infty}} \leq 1$.

Under our point of view in this paper, both the above conditions of mixed multiplicity systems of N are necessary to characterize mixed multiplicities of ideals in terms of the Euler-Poincare characteristic and the mixed multiplicity symbol of $\mathcal{N}$ with respect to a mixed multiplicity system of $\mathcal{N}$.

Remark 4.5. Remark 4.3 proves that if $\mathbf{x}$ is a Rees's superficial sequence of N with respect to $J, \mathbf{I}$ and $\mathbf{x}^*$ is the image of $\mathbf{x}$ in $\bigcup_{i=0}^{d} T_i$, then $\mathbf{x}^*$ is a mixed multiplicity system of $\mathcal{N}$ if and only if $\mathbf{x}$ is a mixed multiplicity system of N with respect to $J, \mathbf{I}$.

In order to describe mixed multiplicity formulas of ideals we need to extend the notion of mixed multiplicities.

Definition 4.6. Let $k_0 \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^d$ such that $k_0 + \mid \mathbf{k} \mid \geq \dim \dfrac{N}{0_N : I^{\infty}} - 1$. We assign

$$E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \begin{cases} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) & \text{if } k_0 + \mid \mathbf{k} \mid = \dim \dfrac{N}{0_N : I^{\infty}} - 1, \\ 0 & \text{if } k_0 + \mid \mathbf{k} \mid > \dim \dfrac{N}{0_N : I^{\infty}} - 1. \end{cases}$$

Note that since

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) := e(\mathcal{N}; k_0, \mathbf{k}),$$

it follows that $E(\mathcal{N}; h_0, \mathbf{h}) = E(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; N)$ for all $h_0 + \mid \mathbf{h} \mid \geq \dim \operatorname{Supp}_{++}\mathcal{N}$.

Put $I_0 = J$. Next, assume that $a \in I_i$ is a Rees's superficial element of N with respect to $J, \mathbf{I}$ and a^* is the image of a in T_i. Then we have

$$\begin{aligned}
&(J^{n+1}\mathbb{I}^{\mathbf{n}}I_iN : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N\\
&= \big((J^{n+1}\mathbb{I}^{\mathbf{n}}I_iN\bigcap aN) : a\big)\bigcap J^n\mathbb{I}^{\mathbf{n}}N\\
&= \big(aJ^{n+1}\mathbb{I}^{\mathbf{n}}N : a\big)\bigcap J^n\mathbb{I}^{\mathbf{n}}N\\
&= \big(J^{n+1}\mathbb{I}^{\mathbf{n}}N + (0_N : a)\big)\bigcap J^n\mathbb{I}^{\mathbf{n}}N\\
&= J^{n+1}\mathbb{I}^{\mathbf{n}}N + (0_N : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N
\end{aligned}$$

for all $n \gg 0;\ \mathbf{n} \gg \mathbf{0}$. Consequently

$$(J^{n+1}\mathbb{I}^{\mathbf{n}}I_iN : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N = J^{n+1}\mathbb{I}^{\mathbf{n}}N + (0_N : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N \tag{4.2}$$

for all $n \gg 0;\ \mathbf{n} \gg \mathbf{0}$. From (4.2) it follows that

$$\begin{aligned}
(0_{\mathcal{N}} : a^*)_{(n,\ \mathbf{n})} &= \frac{(J^{n+1}\mathbb{I}^{\mathbf{n}}I_iN : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N}{J^{n+1}\mathbb{I}^{\mathbf{n}}N}\\
&= \frac{J^{n+1}\mathbb{I}^{\mathbf{n}}N + (0_N : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N}{J^{n+1}\mathbb{I}^{\mathbf{n}}N}\\
&= \frac{(0_N : a)\bigcap J^n\mathbb{I}^{\mathbf{n}}N}{(0_N : a)\bigcap J^{n+1}\mathbb{I}^{\mathbf{n}}N}
\end{aligned}$$

for all $n \gg 0;\ \mathbf{n} \gg \mathbf{0}$. Remember that $I = JI_1 \cdots I_d$. By Artin-Rees lemma, there exists $u \gg 0$ such that

$$(0_{\mathcal{N}} : a^*)_{(n+u,\ \mathbf{n}+u\mathbf{1})} = \frac{[(0_N : a)\bigcap I^uN]J^n\mathbb{I}^{\mathbf{n}}}{[(0_N : a)\bigcap I^uN]J^{n+1}\mathbb{I}^{\mathbf{n}}} \tag{4.3}$$

for all $n \geq 0;\ \mathbf{n} \geq \mathbf{0}$. Fix the u and set

$$W = (0_N : a)\bigcap I^uN;\ U = (0_N : a)/W.$$

Since

$$W : I^\infty = 0_N : aI^\infty \supseteq 0_N : a,$$

it follows that $U/0_U : I^\infty = 0$. So $\dim U/0_U : I^\infty < 0$. Hence since the exact sequence

$$0 \longrightarrow W \longrightarrow (0_N : a) \longrightarrow U = (0_N : a)/W \longrightarrow 0,$$

we get that the mixed multiplicities of $(0_N : a)$ and W with respect to ideals $J, \mathbf{I}$ are the same by [33, Corollary 3.9 (ii)]. So

$$E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; 0_N : a).$$

On the other hand by (4.3), we have

$$E(0_{\mathcal{N}} : a^*; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W).$$

Hence

$$E(0_{\mathcal{N}} : a^*; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; 0_N : a).$$

By (4.1), we get

$$E(\mathcal{N}/a^*\mathcal{N}; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N/aN).$$

The above facts yield:

Remark 4.7. We have $E(\mathcal{N}; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$. And if $a \in I_i$ is a Rees's superficial element of N with respect to $J, \mathbf{I}$ and a^* is the image of a in T_i then

$$E(0_{\mathcal{N}} : a^*; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; 0_N : a);$$
$$E(\mathcal{N}/a^*\mathcal{N}; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N/aN).$$

Now let $\mathbf{x}$ be a mixed multiplicity system of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ and let $\mathbf{x}^*$ be the image of $\mathbf{x}$ in $\bigcup_{i=0}^{d} T_i$. Then $\mathbf{x}^*$ is a mixed multiplicity system of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ by Remark 4.5. Hence $\widetilde{e}(\mathbf{x}^*, \mathcal{N}) = \chi(\mathbf{x}^*, \mathcal{N})$ by Proposition 3.8. And if $k_0 + \mid \mathbf{k} \mid \geq \dim \operatorname{Supp}_{++}\mathcal{N}$ then

$$\widetilde{e}(\mathbf{x}^*, \mathcal{N}) = \chi(\mathbf{x}^*, \mathcal{N}) = E(\mathcal{N}; k_0, \mathbf{k})$$

by Theorem 3.12. On the one hand $E(\mathcal{N}; k_0, \mathbf{k}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$ by Remark 4.7. On the other hand $\dim \operatorname{Supp}_{++}\mathcal{N} = \dim \dfrac{N}{0_N : I^\infty} - 1$ by Remark 4.1. Therefore,

$$\widetilde{e}(\mathbf{x}^*, \mathcal{N}) = \chi(\mathbf{x}^*, \mathcal{N}) = E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$$

if $k_0 + \mid \mathbf{k} \mid \geq \dim \dfrac{N}{0_N : I^\infty} - 1$. Consequently, we have:

Corollary 4.8. *Let $\mathbf{x}$ be a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ and let $\mathbf{x}^*$ be the image of $\mathbf{x}$ in $\bigcup_{i=0}^{d} T_i$. Then the following statements hold.*

(i) $\mathbf{x}^*$ *is a mixed multiplicity system of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$.*

(ii) $\chi(\mathbf{x}^*, \mathcal{N}) = \widetilde{e}(\mathbf{x}^*, \mathcal{N})$.

(iii) *If $k_0 + \mid \mathbf{k} \mid \geq \dim \dfrac{N}{0_N : I^\infty} - 1$ then $E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \chi(\mathbf{x}^*, \mathcal{N}) = \widetilde{e}(\mathbf{x}^*, \mathcal{N})$.*

In the case that $JI_1 \cdots I_d$ is not contained in $\sqrt{\mathrm{Ann}_R N}$, by Corollary 4.8 we immediately get the main result of this section.

Theorem 4.9. *Let $(R, \mathfrak{n})$ denote a noetherian local ring with maximal ideal $\mathfrak{n}$ and infinite residue field $R/\mathfrak{n}$. Let $J, I_1, \ldots, I_d$ be ideals of R with J being $\mathfrak{n}$-primary. Let N be a finitely generated R-module. Assume that $I = JI_1 \cdots I_d$ is not contained in $\sqrt{\mathrm{Ann}_R N}$ and $(k_0, \mathbf{k}) \in \mathbb{N}^{d+1}$ such that $k_0 + \mid \mathbf{k} \mid = \dim \dfrac{N}{0_N : I^\infty} - 1$. Let $\mathbf{x}$ be a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ and $\mathbf{x}^*$ the image of $\mathbf{x}$ in $\bigcup_{i=0}^{d} T_i$. Then we have*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \chi(\mathbf{x}^*, \mathcal{N}) = \widetilde{e}(\mathbf{x}^*, \mathcal{N}).$$

Using different sequences, one expressed mixed multiplicities of ideals in terms of the Samuel multiplicity. For instance: in the case of $\mathfrak{n}$-primary ideals, Risler-Teissier [20] in 1973 showed that each mixed multiplicity is the multiplicity of an ideal generated by a superficial sequence and Rees [15] in 1984 proved that mixed multiplicities are multiplicities of ideals generated by joint reductions; for the case of arbitrary ideals, Viet [25] in 2000 characterized mixed multiplicities as the Samuel multiplicity via (FC)-sequences.

Definition 4.10 ([25])**.** Let $(R, \mathfrak{n})$ denote a noetherian local ring with maximal ideal $\mathfrak{n}$ and infinite residue field $R/\mathfrak{n}$. Let N be a finitely generated R-module. Let $\mathbf{I} = I_1, \ldots, I_d$ be ideals of R. Set $\mathfrak{I} = I_1 \cdots I_d$. An element $a \in R$ is called a *weak-(FC)- element of N with respect to* $\mathbf{I}$ if there exists $i \in \{1, \ldots, d\}$ such that $a \in I_i$ and the following conditions are satisfied:

(i) a is an $\mathfrak{I}$-filter-regular element with respect to N, i.e., $0_N : a \subseteq 0_N : \mathfrak{I}^\infty$.

(ii) a is a Rees's superficial element of N with respect to $\mathbf{I}$.

Let $x_1, \ldots, x_t$ be a sequence in R. Then $x_1, \ldots, x_t$ is called a *weak-(FC)-sequence of N with respect to* $\mathbf{I}$ if x_{i+1} is a weak-(FC)-element of $\dfrac{N}{(x_1, \ldots, x_i)N}$ with respect to $\mathbf{I}$ for all $i = 0, \ldots, t-1$. A weak-(FC)-sequence of N consisting of k_1 elements of $I_1, \ldots, k_d$ elements of I_d is called a weak-(FC)-sequence of N of the type $\mathbf{k} = (k_1, \ldots, k_d)$.

Note that [25] defined weak-(FC)-sequences in the condition $\mathfrak{I} \nsubseteq \sqrt{\mathrm{Ann}_R N}$ (see e.g. [5, 7, 14, 26, 27, 28, 30, 32, 33]). In Definition 4.10, we omitted this condition.

The following comments are very useful for stating the next results.

Remark 4.11. Keep the notations as in Theorem 4.9. Set $J = I_0$ and $I = JI_1 \cdots I_d$. Denote by z^* the image of z in $\bigcup_{i=0}^{d} T_i$ for each $z \subseteq \bigcup_{i=0}^{d} I_i$.

(i) By Remark 4.1, $I \subseteq \sqrt{\mathrm{Ann}_R N}$ if and only if $\dim \mathrm{Supp}_{++}\mathcal{N} < 0$. In this case, $\mathbf{x}$ is a weak-(FC)-sequence and $\mathbf{x}^*$ is a T_{++}-filter-regular sequence with respect to $\mathcal{N}$ for all $\mathbf{x} \subseteq \bigcup_{i=0}^{d} I_i$. On the other hand, if $I \nsubseteq \sqrt{\mathrm{Ann}_R N}$ then [33, Proposition 4.5] proved that if $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to $J, \mathbf{I}$ then $\mathbf{x}^*$ is a T_{++}-filter-regular sequence with respect to $\mathcal{N}$. Hence if $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to $J, \mathbf{I}$ then $\mathbf{x}^*$ is a T_{++}-filter-regular sequence with respect to $\mathcal{N}$.

(ii) Assume that a is a weak-(FC)-element of N and a^* is the image of a in T. Then by (i), a^* is a T_{++}-filter-regular element with respect to $\mathcal{N}$. Hence

$$\dim \mathrm{Supp}_{++}\frac{\mathcal{N}}{a^*\mathcal{N}} \leq \dim \mathrm{Supp}_{++}\mathcal{N} - 1$$

by Remark 2.5. Now remember that $\dim \mathrm{Supp}_{++}\mathcal{N} = \dim \dfrac{N}{0_N : I^\infty} - 1$ and $\dim \mathrm{Supp}_{++}\frac{\mathcal{N}}{a^*\mathcal{N}} = \dim \dfrac{N}{aN : I^\infty} - 1$ by Remark 4.1. Consequently,

$$\dim \frac{N}{aN : I^\infty} \leq \dim \frac{N}{0_N : I^\infty} - 1.$$

On the one hand if $I \subseteq \sqrt{\mathrm{Ann}_R N}$ then $\mathbf{x}$ is a weak-(FC)-sequence for all $\mathbf{x} \subseteq \bigcup_{i=0}^{d} I_i$. On the other hand if $I \nsubseteq \sqrt{\mathrm{Ann}_R N}$ then for any $0 \leq i \leq d$, there exists a weak-(FC)-element of N with respect to $J, \mathbf{I}$ by [14, Proposition 2.3]. From this it follows that for each $(k_0, \mathbf{k}) \in \mathbb{N}^{d+1}$, there exists a weak-$(FC)$-sequence $\mathbf{x}$ of the type $(k_0, \mathbf{k})$. Now we choose $(k_0, \mathbf{k})$ such that

$$k_0 + \mid \mathbf{k} \mid \geq \dim \frac{N}{0_N : I^\infty} - 1,$$

then $\dim \dfrac{N}{\mathbf{x}N : I^\infty} \leq 1$. Hence $\mathbf{x}$ is a mixed multiplicity system of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$. So for any $(k_0, \mathbf{k}) \in \mathbb{N}^{d+1}$ with

$$k_0 + \mid \mathbf{k} \mid \geq \dim \frac{N}{0_N : I^\infty} - 1,$$

there exists a mixed multiplicity system of N of the type $(k_0, \mathbf{k})$.

(iii) Note that if $I \nsubseteq \sqrt{\mathrm{Ann}_R N}$, then $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{0}]}; N) = e(J; \frac{N}{0_N : I^\infty})$ by [25, Proposition 3.2].

Corollary 4.12. *Let* $k_0+ \mid \mathbf{k} \mid= \dim \dfrac{N}{0_N : I^\infty} - 1$. *Let* $\mathbf{x}$ *be a weak-*(FC)*-sequence of* N *with respect to* $J, \mathbf{I}$ *of the type* $(k_0, \mathbf{k})$ *and let* $\mathbf{x}^*$ *be the image of* $\mathbf{x}$ *in* $\bigcup_{i=0}^d T_i$. *Then*

(i) $\widetilde{e}(\mathbf{x}^*, \mathcal{N}) = \chi(\mathbf{x}^*, \mathcal{N}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = E(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N})$.

(ii) $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$ *if and only if* $\dim \frac{N}{\mathbf{x}N : I^\infty} = 1$. *In this case,*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J; \frac{N}{\mathbf{x}N : I^\infty}).$$

Proof. Since $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$, $\mathbf{x}^*$ is a T_{++}-filter-regular sequence with respect to $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ by Remark 4.11(i). On the other hand $k_0+ \mid \mathbf{k} \mid= \dim \dfrac{N}{0_N : I^\infty} - 1$, $\dim \operatorname{Supp}_{++}\mathcal{N} = k_0+ \mid \mathbf{k} \mid$ by Remark 4.1. So $E(\mathcal{N}; k_0, \mathbf{k}) = e(\mathcal{N}; k_0, \mathbf{k})$. Hence by Corollary 3.18, we have

$$e(\mathcal{N}; k_0, \mathbf{k}) = E(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}).$$

Remember that $E(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}) = E(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N/\mathbf{x}N)$ by Remark 4.7 and

$$e(\mathcal{N}; k_0, \mathbf{k}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N).$$

Consequently, by Theorem 4.9 we get (i) that

$$\widetilde{e}(\mathbf{x}^*, \mathcal{N}) = \chi(\mathbf{x}^*, \mathcal{N}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = E(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}).$$

So $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$ if and only if $E(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}) \neq 0$. This is equivalent to $\dim \frac{N}{\mathbf{x}N : I^\infty} = 1$ and $e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}) \neq 0$. But if $\dim \frac{N}{\mathbf{x}N : I^\infty} = 1$ then

$$e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}) = e(J; \frac{N}{\mathbf{x}N : I^\infty}) \neq 0$$

by Remark 4.11(iii). Hence $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$ if and only if $\dim \frac{N}{\mathbf{x}N : I^\infty} = 1$. □

Recall that [25, Theorem 3.4] showed that (see e.g. [6, 7, 14, 27, 30, 32])

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$$

if and only if there exists a weak-(FC)*-sequence of* N *of the type* $(0, \mathbf{k})$ *with respect to* $J, \mathbf{I}$ *such that* $\dim N/\mathbf{x}N : I^\infty = \dim N/0_N : I^\infty - \mid \mathbf{k} \mid$. *In this case, we get*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \frac{N}{\mathbf{x}N : I^\infty}\big).$$

Hence Corollary 4.12(ii) is also an immediate consequence of [25, Theorem 3.4].

By Remark 4.11(ii), for each $(k_0, \mathbf{k}) \in \mathbb{N}^{d+1}$ with $k_0 + \mid \mathbf{k} \mid \geq \dim \dfrac{N}{0_N : I^\infty} - 1$, there exists a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$. Then by Corollary 3.19 and Remark 4.7, we obtain the following corollary.

Corollary 4.13. *Assume that $I = JI_1 \cdots I_d$ is not contained in $\sqrt{\mathrm{Ann}_R N}$ and $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}, N) \neq 0$. Let $\mathbf{x} = x_1, \ldots, x_s$ be a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$. Denote by $(m_i, \mathbf{h}_i) = (m_i, h_{i1}, \ldots, h_{id})$ the type of a subsequence $x_1, \ldots, x_i$ of $\mathbf{x}$ for each $1 \leq i \leq s$. And for each $1 \leq i \leq s$, set*

$$N_i = \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N}.$$

Then the following statements hold.

(i) $\dim \dfrac{N}{(x_1, \ldots, x_i)N : I^\infty} = \dim \dfrac{N}{0_N : I^\infty} - i$ *for each* $1 \leq i \leq s$.

(ii) $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \dfrac{N}{\mathbf{x}N : I^\infty}\big) - \sum_{i=1}^{s} E\big(J^{[k_0 - m_i + 1]}, \mathbf{I}^{[\mathbf{k} - \mathbf{h}_i]}; N_i\big)$.

Proof. Since $\mathbf{x}$ is a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$, $\mathbf{x}^*$ is a mixed multiplicity system of the type $(k_0, \mathbf{k})$ of $\mathcal{N}$ by Remark 4.5. Recall that

$$e(\mathcal{N}; k_0, \mathbf{k}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N).$$

Hence $e(\mathcal{N}; k_0, \mathbf{k}) \neq 0$ since $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$. Consequently by Corollary 3.19(i), we have

$$\dim \mathrm{Supp}_{++}\big[\frac{\mathcal{N}}{(x_1^*, \ldots, x_i^*)\mathcal{N}}\big] = \dim \mathrm{Supp}_{++}\mathcal{N} - i$$

for each $1 \leq i \leq s$. Therefore

$$\dim \frac{N}{(x_1, \ldots, x_i)N : I^\infty} = \dim \frac{N}{0_N : I^\infty} - i$$

for each $1 \leq i \leq s$ by Remark 4.1. We get (i). Remember that

$$e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}) = e(J; \frac{N}{\mathbf{x}N : I^\infty})$$

by Remark 4.11(iii). On the one hand

$$E(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}) = E(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N})$$

by Remark 4.7. On the other hand since

$$\dim \operatorname{Supp}_{++}\big[\frac{\mathcal{N}}{\mathbf{x}^*\mathcal{N}}\big] = 0 \ \text{ and } \ \dim \frac{N}{\mathbf{x}N : I^\infty} = 1,$$

we get

$$E(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}) = e(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}) \text{ and } E(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}) = e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; \frac{N}{\mathbf{x}N}).$$

Hence

$$e(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}) = e(J; \frac{N}{\mathbf{x}N : I^\infty}).$$

It is easily seen by Remark 4.7 that for each $1 \le i \le s$,

$$\begin{aligned} &E\bigg(\frac{(x_1^*, \ldots, x_{i-1}^*)\mathcal{N} : x_i^*}{(x_1^*, \ldots, x_{i-1}^*)\mathcal{N}}; k_0 - m_i, \mathbf{k} - \mathbf{h}_i\bigg) \\ &= E\bigg(J^{[k_0 - m_i + 1]}, \mathbf{I}^{[\mathbf{k} - \mathbf{h}_i]}; \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N}\bigg). \end{aligned}$$

Consequently, by Corollary 3.19(ii) we get (ii). $\square$

In particular, if $\mathbf{I} = I_1, \ldots, I_d$ are $\mathfrak{n}$-primary ideals, then $\dim \frac{N}{0_N : I^\infty} = \dim N$ and $\dim \frac{N}{(x_1, \ldots, x_i)N} = \dim \frac{N}{(x_1, \ldots, x_i)N : I^\infty}$ for each $1 \le i \le s$. Hence we get

$$\dim \frac{N}{(x_1, \ldots, x_i)N} = \dim N - i$$

for each $1 \le i \le s$. So $x_1, \ldots, x_s$ is a part of a parameter system for N. Moreover, in this case, $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}, N) \ne 0$ for each $(k_0, \mathbf{k})$ with $k_0 + \mid \mathbf{k} \mid = \dim N - 1$ by [20]; and $e\big(J; \frac{N}{\mathbf{x}N : I^\infty}\big) = e\big(J; \frac{N}{\mathbf{x}N}\big)$ by [25].

Then as an immediate consequence of Corollary 4.13, we have the following result.

Corollary 4.14. *Let $J; \mathbf{I}$ be $\mathfrak{n}$-primary ideals and $\dim N > 0$. Let $\mathbf{x} = x_1, \ldots, x_s$ be a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$. Denote by $(m_i, \mathbf{h}_i) = (m_i, h_{i1}, \ldots, h_{id})$ the type of a subsequence $x_1, \ldots, x_i$ of $\mathbf{x}$ for each $1 \le i \le s$. And put*

$$N_i = \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N}$$

for each $1 \le i \le s$. Then we have the following statements.

(i) $\mathbf{x}$ *is a part of a parameter system for* N.

(ii) $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \frac{N}{\mathbf{x}N}\big) - \sum_{i=1}^{s} E\big(J^{[k_0 - m_i + 1]}, \mathbf{I}^{[\mathbf{k} - \mathbf{h}_i]}; N_i\big).$

References

[1] M. Auslander and D. A. Buchsbaum, *Codimension and multiplicity*, Ann. Math. 68(1958), 625-657.

[2] M. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge studies in Advanced Mathematics, **60**, Cambridge, Cambridge University Press, 1998.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, **39**, Cambridge, Cambridge University Press, 1993.

[4] R. Callejas-Bedregal and V. H. Jorge Perez, *Mixed multiplicities and the minimal number of generator of modules*, J. Pure Appl. Algebra 214 (2010), 1642-1653.

[5] R. Callejas-Bedregal and V. H. Jorge Perez, (FC)-*Sequences, mixed multiplicities and reductions of modules*, arXiv:1109.5058(2011).

[6] L. V. Dinh, N. T. Manh, T. T. H. Thanh, *A Survey of some results for mixed multiplicities*, arXiv:1207.2850(2012).

[7] L. V. Dinh and D. Q. Viet, *On two results of mixed multiplicities*, Int. J. Algebra 4(1) 2010, 19-23.

[8] M. Herrmann, E. Hyry, J. Ribbe, Z. Tang, *Reduction numbers and multiplicities of multigraded structures*, J. Algebra 197(1997), 311-341.

[9] E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc. 351(1999), 2213-2232.

[10] D. Katz, J. K. Verma, *Extended Rees algebras and mixed multiplicities*, Math. Z. 202(1989), 111-128.

[11] D. Kirby and D. Rees, *Multiplicities in graded rings I: the general theory*, Contemporary Mathematics 159(1994), 209 - 267.

[12] D. Kirby and D. Rees, *Multiplicities in graded rings II: integral equivalence and the Buchsbaum - Rim multiplicity*, Math. Proc. Cambridge Phil. Soc. 119 (1996), 425 - 445.

[13] S. Kleiman and A. Thorup, *Mixed Buchsbaum - Rim multiplicities*, Amer. J. Math. 118(1996), 529-569.

[14] N. T. Manh and D. Q. Viet, *Mixed multiplicities of modules over Noetherian local rings*, Tokyo J. Math. Vol. 29. No. 2, (2006), 325-345.

[15] D. Rees, *Generalizations of reductions and mixed multiplicities*, J. London. Math. Soc. 29(1984), 397-414.

[16] P. Roberts, *Local Chern classes, multiplicities and perfect complexes*, Memoire Soc. Math. France 38(1989), 145 - 161.

[17] J. P. Serre, *Algebre locale. Multiplicities*. LNM 11, Springer, 1965.

[18] J. Stuckrad and W. Vogel, *Buchsbaum rings and applications*, VEB Deutscher Verlag der Wisssenschaften. Berlin, 1986.

[19] I. Swanson, *Mixed multiplicities, joint reductions and quasi-unmixed local rings* , J. London Math. Soc. 48(1993), no. 1, 1 - 14.

[20] B. Teisier, *Cycles èvanescents, sections planes, et conditions de Whitney*, Singularities à Cargése, 1972. Astérisque, 7-8(1973), 285-362.

[21] N. V. Trung, *Reduction exponents and degree bound for the defining equation of graded rings*, Proc. Amer. Mat. Soc. 101(1987), 229-234.

[22] N. V. Trung, *Positivity of mixed multiplicities*, J. Math. Ann. 319(2001), 33 - 63.

[23] N. V. Trung and J. Verma, *Mixed multiplicities of ideals versus mixed volumes of polytopes*, Trans. Amer. Math. Soc. 359(2007), 4711-4727.

[24] J. K. Verma, *Multigraded Rees algebras and mixed multiplicities*, J. Pure and Appl. Algebra 77(1992), 219-228.

[25] D. Q. Viet, *Mixed multiplicities of arbitrary ideals in local rings*, Comm. Algebra. 28(8)(2000), 3803-3821.

[26] D. Q. Viet, *On some properties of (FC)-sequences of ideals in local rings*, Proc. Amer. Math. Soc. 131 (2003), 45-53.

[27] D. Q. Viet, *Sequences determining mixed multiplicities and reductions of ideals*, Comm. Algebra. 31(2003), 5047-5069.

[28] D. Q. Viet, *Reductions and mixed multiplicities of ideals*, Comm. Algebra. 32(2004), 4159-4178.

[29] D. Q. Viet, *The multiplicity and the Cohen-Macaulayness of extended Rees algebras of equimultiple ideals*, J. Pure and Appl. Algebra 205(2006),498-509.

[30] D. Q. Viet, L. V. Dinh, T. T. H. Thanh, *A note on joint reductions and mixed multiplicities*, arXiv:1110.6239(2011), accepted by Proc. AMS.

[31] D. Q. Viet and N. T. Manh, *Mixed multiplicities of multigraded modules*, Forum Math. (2011), DOI: 10.1515/forum. 2011. 120.

[32] D. Q. Viet and T. T. H. Thanh, *On (FC)-sequences and mixed multiplicities of multigraded algebras*, Tokyo J. Math. 34 (2011), 185-202.

[33] D. Q. Viet and T. T. H. Thanh, *On some multiplicity and mixed multiplicity formulas*, Forum Math. (2011), DOI: 10.1515/forum. 2011.168.

[34] D. Q. Viet and T. T. H. Thanh, *A note on formulas transmuting mixed multiplicities*, Forum Math. (2012), DOI: 10.1515/forum. 2011. 0147.